テンソルネットワーク法の素粒子物理学への応用

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Plan of Talk

- Introduction
- Tensor Renormalization Group (TRG)
  - Two Dimensional Ising Model
  - Singular Value Decomposition
- Current Status for TRG Studies in Particle Physics
- Application to Finite Density System
  - 2D Complex ϕ^4 Theory at Finite Density
- Summary

Introduction

Partition function of lattice gauge theory

$$Z = \int \mathcal{D}U \; \det D(\{U\})\, \mathrm{e}^{-S_g(\{U\})}$$

U: gauge field
D: Dirac matrix
$\bar{\psi} D \psi$

Expectation value of physical quantity

$$\langle O \rangle = \int \mathcal{D}U \, O(\{U, D^{-1}\}) \; \det D(\{U\})\, \mathrm{e}^{-S_g(\{U\})}$$

Monte Carlo works for $\det D e^{-S} > 0$ with importance sampling

$$P = \frac{1}{Z} \det D(\{U\})\, \mathrm{e}^{-S_g(\{U\})}$$

In case of finite density simulation
Introduction of chemical potential μ ⇒ detD becomes a complex value,
Importance sampling fails = statistical error becomes uncontrollable

Sign problem / Complex action problem

Tensor Network Scheme

What is Tensor Network (TN) Scheme?

Theoretical and numerical methods for high precision analyses of many body problems with tensor network formalism

What is different from conventional methods?

Free from sign problem and complex action problem in Monte Carlo method
Computational cost for L^D system size $\propto D \times \log(L)$
Direct treatment of Grassmann numbers
Direct evaluation of partition function Z itself

↓

Possible applications in particle physics :

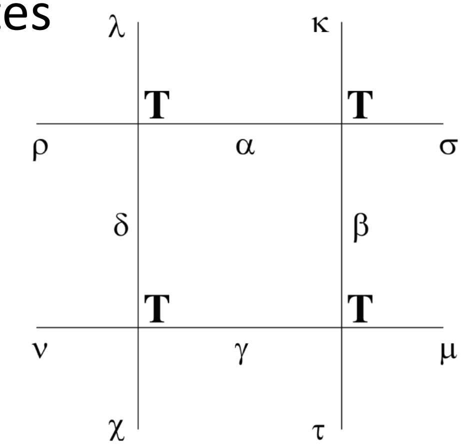
Light quark dynamics in QED/QCD, Finite density QCD,
Strong CP problem, Chiral gauge theories, Lattice SUSY etc.

Also many applications in condensed matter physics

Tensor Renormalization Group (TRG)

Levin-Nave
PRL99(2007)120601

Explain the algorithm with 2D Ising model with N sites



Hamiltonian $H = \sum_{\langle i,j \rangle} s_i s_j \qquad s_i \pm 1$

Partition Function $Z = \sum_{\{s_i\}} \exp\left(-\beta H\right)$

$$= \boxed{\sum_{\alpha,\beta,\gamma,\delta,\cdots=1}^{2} T_{\alpha,\lambda,\rho,\delta} T_{\sigma,\kappa,\alpha,\beta} T_{\mu,\beta,\gamma,\tau} T_{\gamma,\delta,\nu,\chi} \cdots}$$

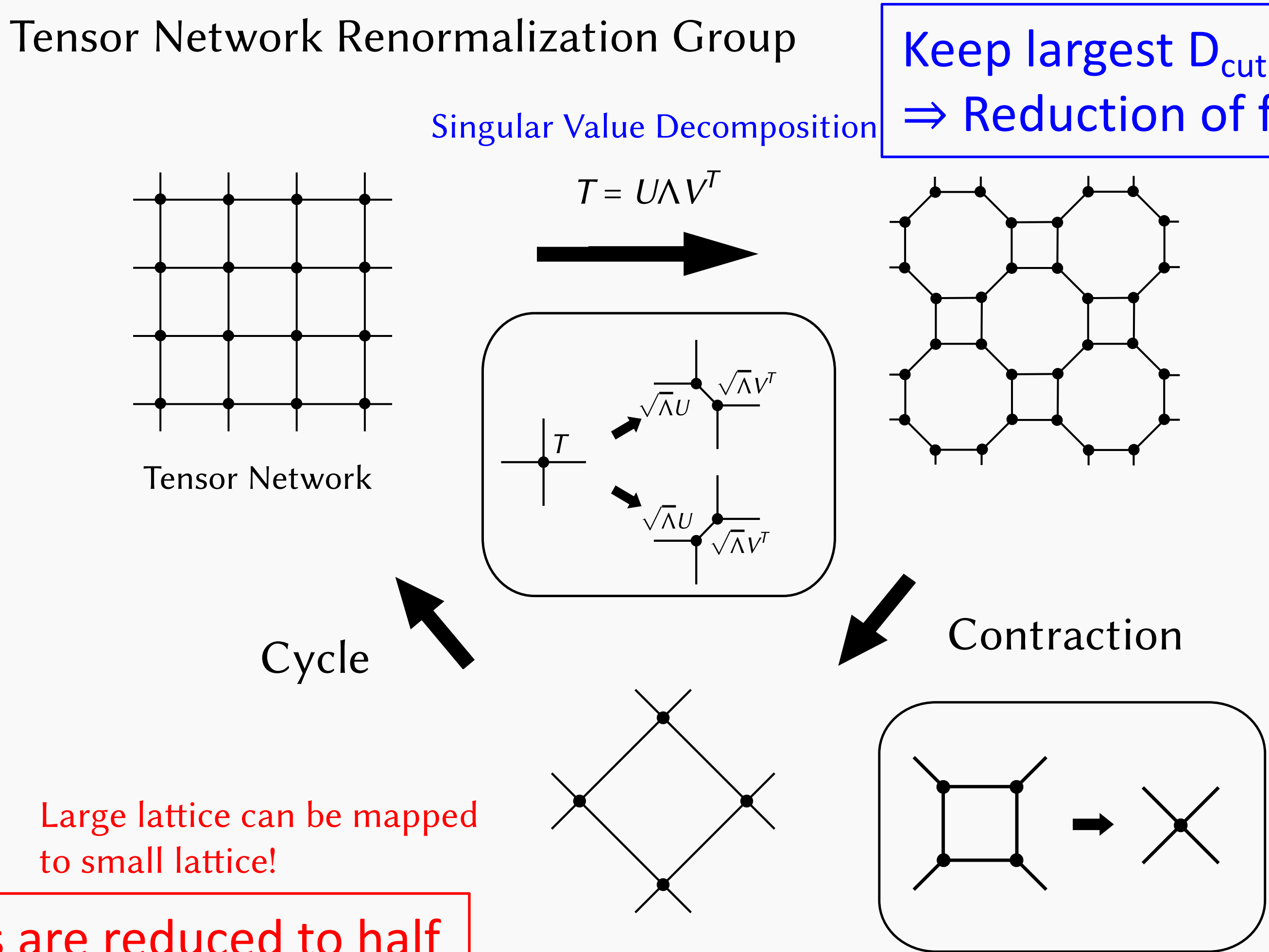
Tensor Network formulation

Details of model are specified in initial tensor
The algorithmic procedure is independent of models

Of course, direct contraction is impossible for large N even with current fastest supercomputer
⇒ How to evaluate the partition function?

Schematic View of TRG Algorithm

1. Singular Value Decomposition of local tensor T
2. Contraction of old tensor indices (coarse-graining)
3. Repeat the iteration



Keep largest D_{cut} components
⇒ Reduction of freedom

Tensors
w/ new indices

#sites are reduced to half

Quick Review of Singular Value Decomposition

Any m × n (m>n) real matrix A can be decomposed as $A=U\Sigma V^T$

U: m × m orthogonal matrix

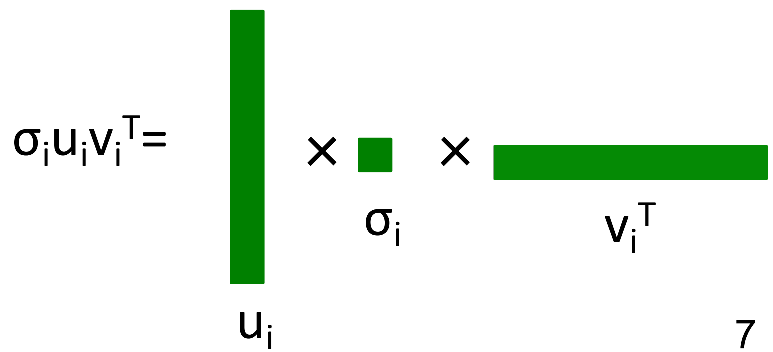
V: n × n orthogonal matrix

$\Sigma=\mathrm{diag}(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \cdots, \sigma_n)$ $(\sigma_1\geq\sigma_2\geq \sigma_3\geq\sigma_4\geq\cdots\geq\sigma_n\geq0)$

$\sigma_1, \sigma_2, \sigma_3, \sigma_4, \cdots, \sigma_n$: Singular Values (SV) of A (non-negative)

Representation with sum of rank=1 matrices using columns $(u_1, u_2, \cdots, u_n)$ in U and $(v_1, v_2, \cdots, v_n)$ in V

$A=\sigma_1 u_1 v_1^T+\sigma_2 u_2 v_2^T+\cdots+\sigma_n u_n v_n^T$



Approximation of Matrix

Truncation of sum of rank 1 matrices

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_k u_k v_k^T + \cdots + \sigma_n u_n v_n^T$$

$$A_k = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_k u_k v_k^T$$

(k-rank approximation with largest k SVs)

Error of approximation:

$$||A - A_k||_F = (\sigma_{k+1}^2 + \sigma_{k+2}^2 + \cdots + \sigma_n^2)^{1/2}$$

where, $||A||_F = (Tr(A^T A))^{1/2} = (\Sigma_i \Sigma_j a_{ij}^2)^{1/2}$

Note: Image compression is a popular application

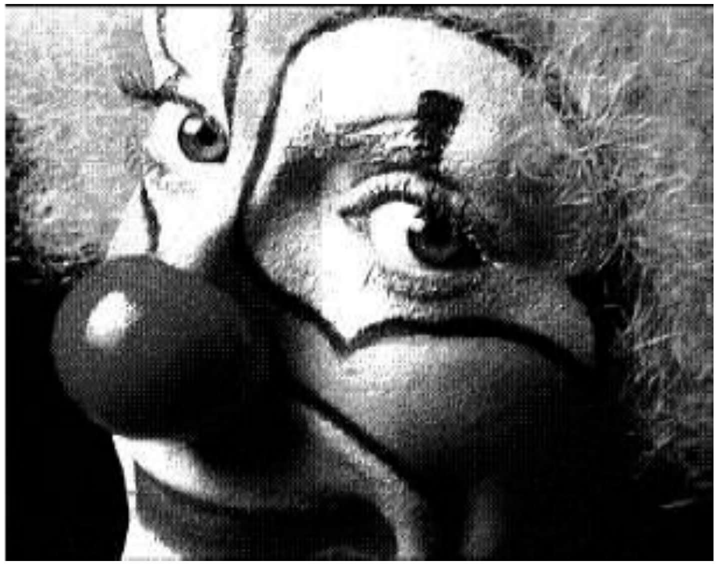
Image Compression with SVD

Image data with 200x320 pixel ⇒ 200x320 real matrix

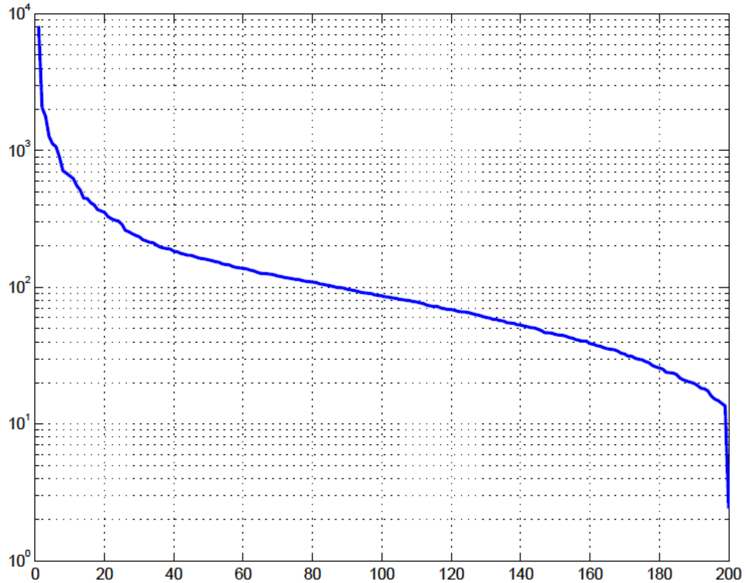
Application of SVD to the matrix

$$A=\sigma_1 u_1 v_1^T+\sigma_2 u_2 v_2^T+\ldots+\sigma_n u_n v_n^T \quad (n=200)$$

Sample image (200x320 pixel)



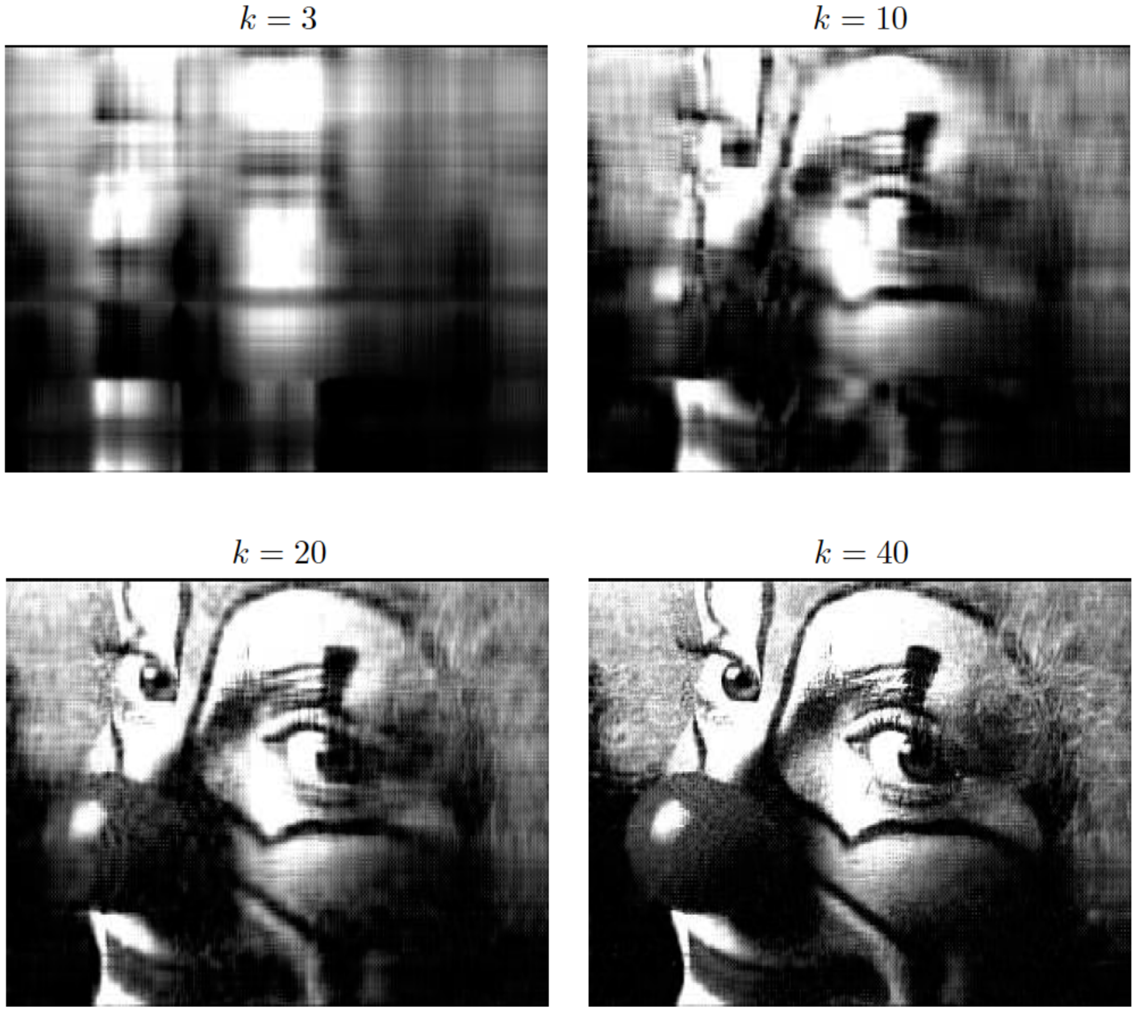
Distribution of SVs



J. Demmel: Applied Numerical Linear Algebra, SIAM 1997

Approximation with SVD

$A_k=\sigma_1 u_1 v_1^T+\sigma_2 u_2 v_2^T+\ldots+\sigma_k u_k v_k^T \quad (k \ll 200)$



J. Demmel: Applied Numerical Linear Algebra, SIAM 1997

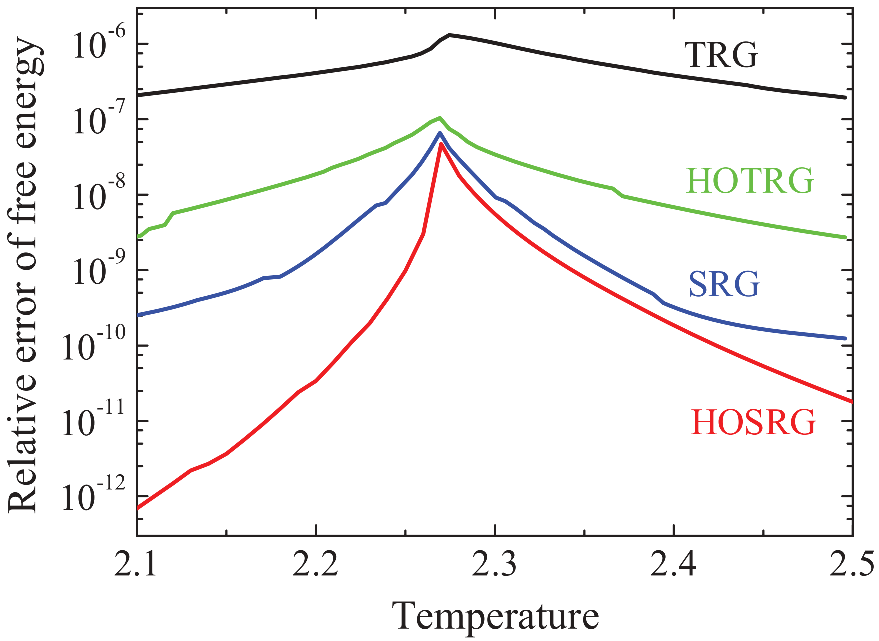
Numerical test for 2D Ising Model

The key element in the algorithm is low-rank approximation by SVD

$$T_{i,j,k,l} \simeq \sum_{m=1}^{\mathrm{D_{cut}}} U_{(i,j),m}\sigma_m V_{m,(k,l)}$$

Truncation error is controlled by the parameter D_{cut}

Free energy on and off the transition point, lattice size=$2^{30\sim50}$, D_{cut}=24



Xie et al.
PRB86(2012)045139

Comparison with analytic results

Relative error of free energy : $\leq 10^{-6}$

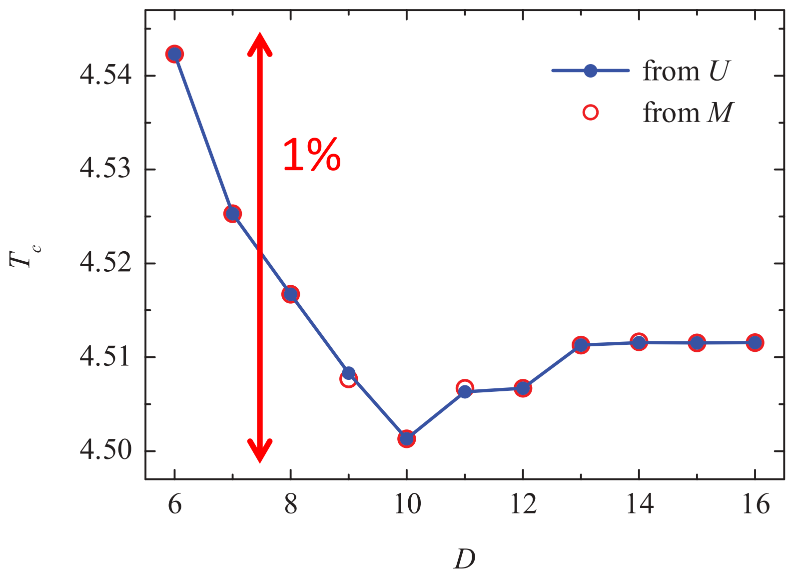
Study of 3D Ising Model

Xie et al.
PRB86(2012)045139

Higher-Order TRG (HOTRG): applicable to higher dimensional models

Computational cost $\propto (D_{cut})^{11} \times \log(V)$

D_{cut} dependence of T_c



Comparison with Monte Carlo data

Method	T_c
HOTRG ($D = 16$, from U)	4.511544
HOTRG ($D = 16$, from M)	4.511546
Monte Carlo[37]	4.511523
Monte Carlo[38]	4.511525
Monte Carlo[39]	4.511516
Monte Carlo[35]	4.511528
Series expansion[40]	4.511536
CTMRG[12]	4.5788
TPVA[13]	4.5704
CTMRG[14]	4.5393
TPVA[16]	4.554
Algebraic variation[41]	4.547

Results show good agreement with Monte Carlo data at high precision

Collaborators

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S. Takeda, R. Sakai	Kanazawa U.
D. Kadoh	N. Tsing-Hua U./ Keio U.

Application of TRGs to Particle Physics (1)

<u>2D models</u>

Ising model : Levin-Nave, PRL99(2007)120601

X-Y model : Meurice+, PRE89(2014)013308

CP(1) : Kawauchi-Takeda, PRD93(2016)114503

Real ϕ^4 theory :

Shimizu, Mod.Phys.Lett.A27(2012)1250035,
Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP1905(2019)184

Complex ϕ^4 theory at finite density :

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, in preparation

U(1) gauge theory+θ :

YK-Yoshimura, arXiv:1911.06480

Schwinger, Schwinger+θ :

Shimizu-YK, PRD90(2014)014508, PRD90(2014)074503,
PRD97(2018)034502

Gross-Neveu model at finite density :

Takeda-Yoshimura, PTEP2015(2015)043B01

N=1 Wess-Zumino model :

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP1803(2018)141

Application of TRGs to Particle Physics (2)

3D models

Ising : Xie+, PRB86(2012)045139

Potts model : Wan+, CPL31(2014)070503

Free Wilson fermion :

Sakai-Takeda-Yoshimura, PTEP2017(2017)063B07,

Yoshimura-YK-Nakamura-Takeda-Sakai, PRD97(2018)054511

Z_2 gauge theory at finite temperature :

YK-Yoshimura, JHEP1908(2019)023

4D models

Ising : Akiyama-YK-Yamashita-Yoshimura, PRD100(2019)054510

weak first-order phase transition (not second-order phase transition)

Application to Finite Density System

2D complex ϕ^4 theory at finite density

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, in preparation

How to treat continuous dof?

Complex action with finite chemical potential μ

Sign problem is really solved?

2D Complex ϕ^4 Theory at Finite Density

Kadoh+, in preparation

Continuum action of 2D complex ϕ^4 theory at finite μ

$$S_{\rm cont} = \int {\rm d}^2 x \left\{ |\partial_\rho \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu(\phi^* \partial_2 \phi - \partial_2 \phi^* \phi) + \lambda |\phi|^4 \right\}$$

Introduction of finite chemical potential ⇒ complex action

Lattice action

$$Z(\text{original}) = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp(-S)$$

$$S = \sum_n \left[(4 + m^2)|\phi_n|^2 + \lambda |\phi_n|^4 - \sum_{\rho=1}^{2} \left(e^{\mu \delta_{\rho,2}} \phi_n^* \phi_{n+\hat{\rho}} + e^{-\mu \delta_{\rho,2}} \phi_{n+\hat{\rho}}^* \phi_n \right) \right]$$

TN representation is constructed in the same way as for the real ϕ^4 case

Bose condensation is expected to occur at sufficiently large μ

Tensor Network Representation

Kadoh+, in preparation

Boltzmann weight is expressed as

$$e^{-S} = \prod_{n\in\Gamma}\prod_{\nu=1}^{2} f_\nu\left(\phi_n, \phi_{n+\hat{\nu}}\right)$$

$$f_\nu\left(z, z'\right)$$
$$= \exp\left\{-\frac{1}{4}\left(4+m^2\right)\left(|z|^2+|z'|^2\right)-\frac{\lambda}{4}\left(|z|^4+|z'|^4\right)+e^{\mu\delta_{\nu,2}}z^*z'+e^{-\mu\delta_{\nu,2}}zz'^*\right\}$$

⇒ Need to discretize the continuous d. o. f.

Use of Gauss-Hermite quadrature

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}z_1\mathrm{d}z_2\; e^{-z_1^2-z_2^2}h\left(z\right) \approx \sum_{\alpha,\beta=1}^{K} w_\alpha w_\beta h\left(\frac{y_\alpha+iy_\beta}{\sqrt{2}}\right)$$

Discretized version of partition function

$$Z \approx Z\left(K\right) = \sum_{\{\alpha,\beta\}}\prod_{n\in\Gamma} w_{\alpha_n}w_{\beta_n}\exp\left(y_{\alpha_n}^2+y_{\beta_n}^2\right)\prod_{\nu=1}^{2} f_\nu\left(\frac{y_{\alpha_n}+iy_{\beta_n}}{\sqrt{2}}, \frac{y_{\alpha_{n+\hat{\nu}}}+iy_{\beta_{n+\hat{\nu}}}}{\sqrt{2}}\right)$$

Simple(st) Test Bed for Sign Problem

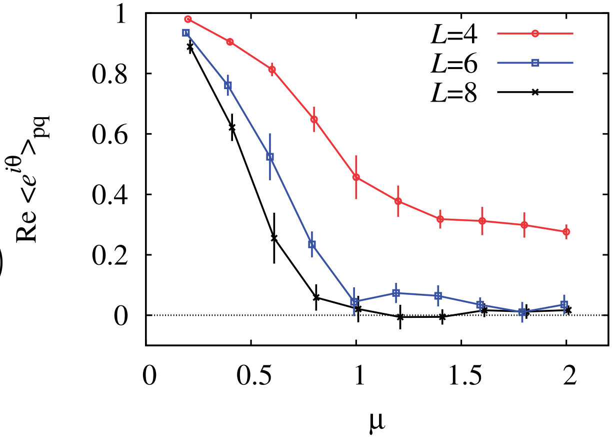
Mori-Kashiwa-Onishi, PTEP(2018)023B04

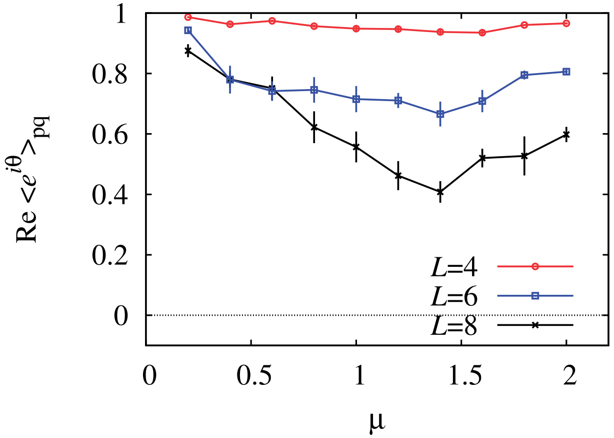
Previous study with path optimization method (Monte Carlo)

Average Phase Factor

$$\langle e^{i\theta} \rangle = Z/Z_{\rm pq}$$

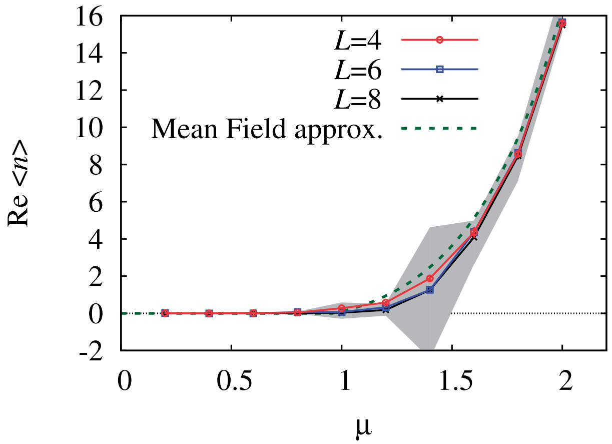
$$Z_{\rm pq} = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp(-{\rm Re}(S))$$





$\langle e^{i\theta} \rangle$ becomes close to 1

Still, it seems difficult to perform a MC simulation on L≳10



$m^2=1$, $\lambda=1$

The authors claim

"We show that the average phase factor is significantly enhanced after the optimization and then we can safely perform the hybrid Monte Carlo method."

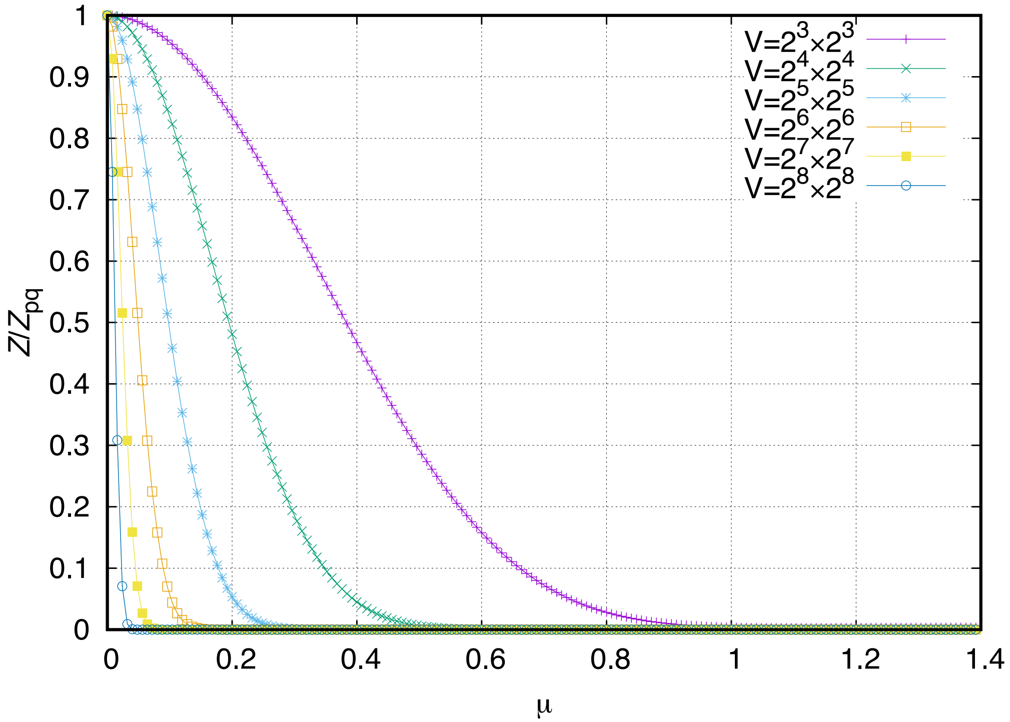
Results for Z(original) with TRG

Kadoh+, in preparation

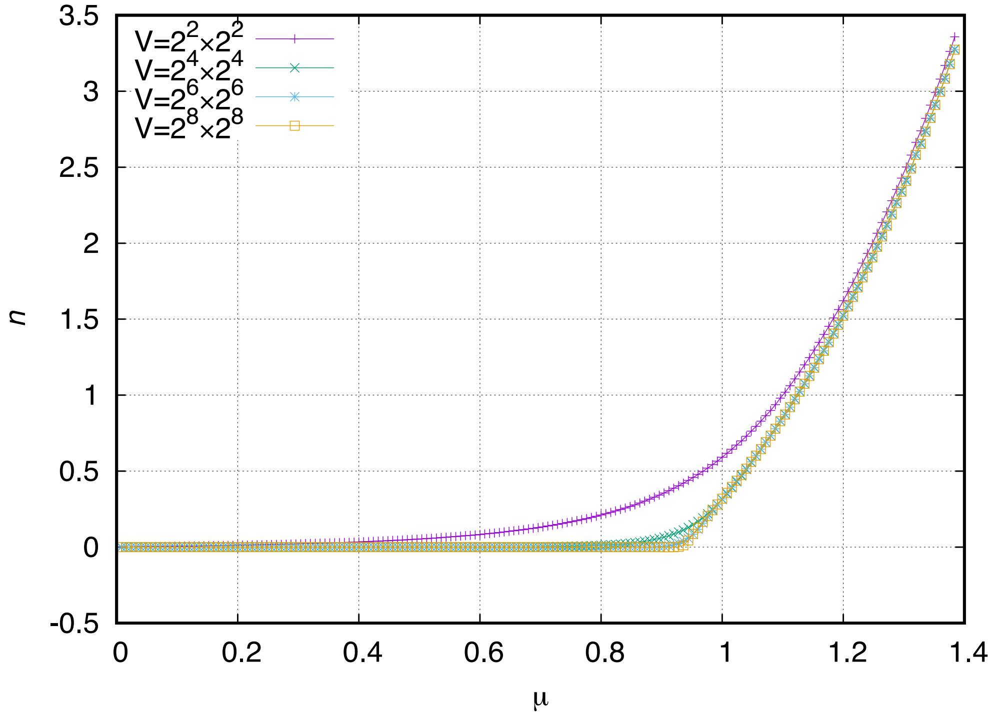
Parameters: m^2=0.01, λ=1, K=64, D_{cut}=64

V=L × L is changed from 4 × 4 to 256 × 256

Average phase factor



Number density



Sign problem: $\langle e^{i\theta}\rangle \ll 1$

Bose condensation@μ≳0.95

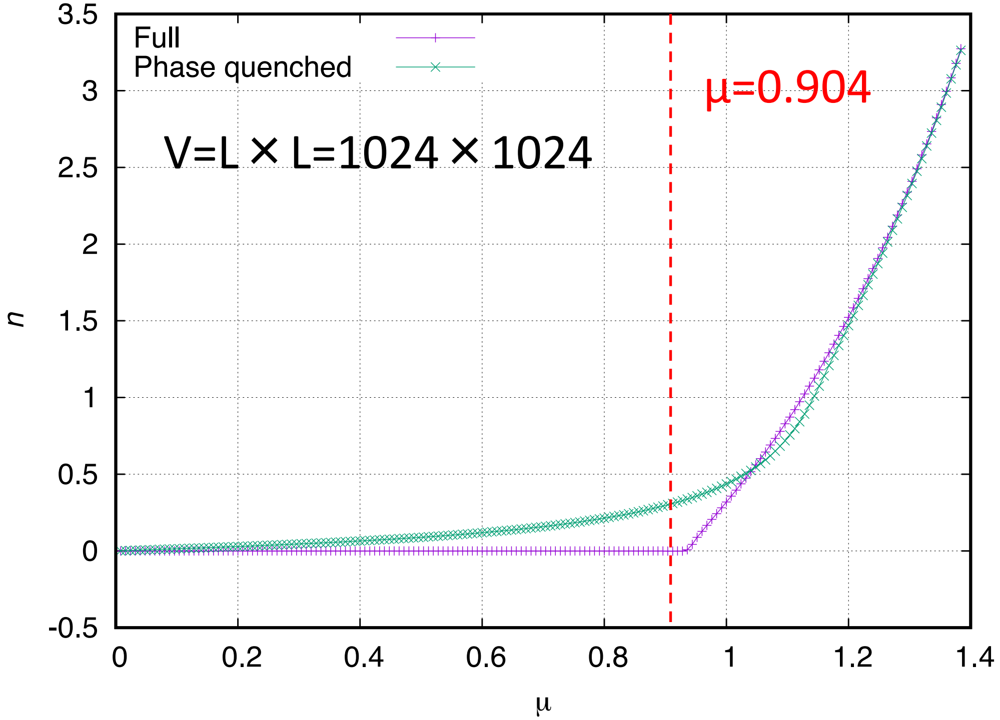
Effects of Phase

Kadoh+, in preparation

Parameters: m^2=0.01, λ=1, K=64, D_{cut}=64

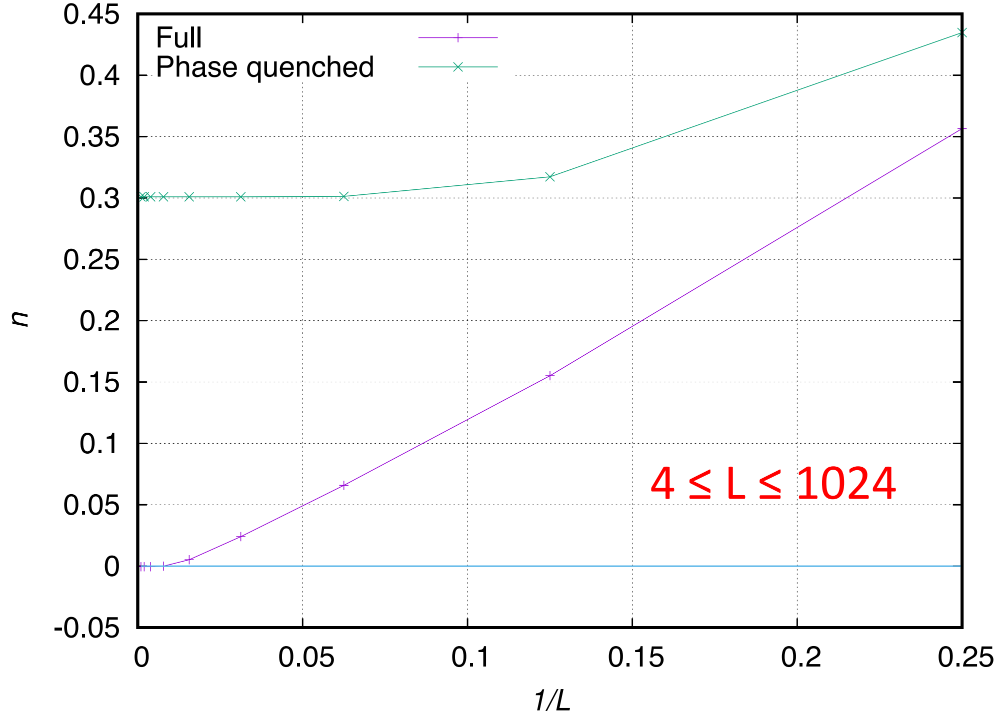
$$Z_{\rm pq} = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp(-{\rm Re}(S))$$

<n> w/ and w/o phase



Clear phase effect

1/L dependence of <n> @ μ=0.904



No Silver Blaze phenomena
in phase quenched case

Sign-Problem-Free Representation

Endres, PRD75(2007)065012

Mathematical tools

- Polar coordinate

$$\phi_n = (\phi_{n,1}, \phi_{n,2}) \rightarrow (r_n \cos\theta_n, r_n \sin\theta_n)$$

- Character expansion

$$\exp(x\cos z) = \sum_{k=-\infty}^{\infty} \mathrm{I}_k(x)\exp(ikz) \quad x \in R,\ z \in C$$

Partition function can be expressed in a sign-problem-free form

$$Z(\text{positive})$$
$$= \left(\prod_n \sum_{k_{n,1},k_{n,2}=-\infty}^{\infty}\right)\left(\prod_n \int_0^\infty \mathrm{d}r_n\right)\prod_n 2\pi r_n \prod_{\rho=1}^{2} e^{-\frac{1}{4}(4+m^2)(r_n^2+r_{n+\hat\rho}^2)-\frac{\lambda}{4}(r_n^4+r_{n+\hat\rho}^4)}$$
$$\cdot \mathrm{I}_{k_{n,\rho}}(2r_n r_{n+\hat\rho}) e^{k_{n,\rho}\mu\delta_{\rho,2}} \delta_{(k_{n,1}+k_{n,2}-k_{n-\hat1,1}-k_{n-\hat2,2}),0}$$

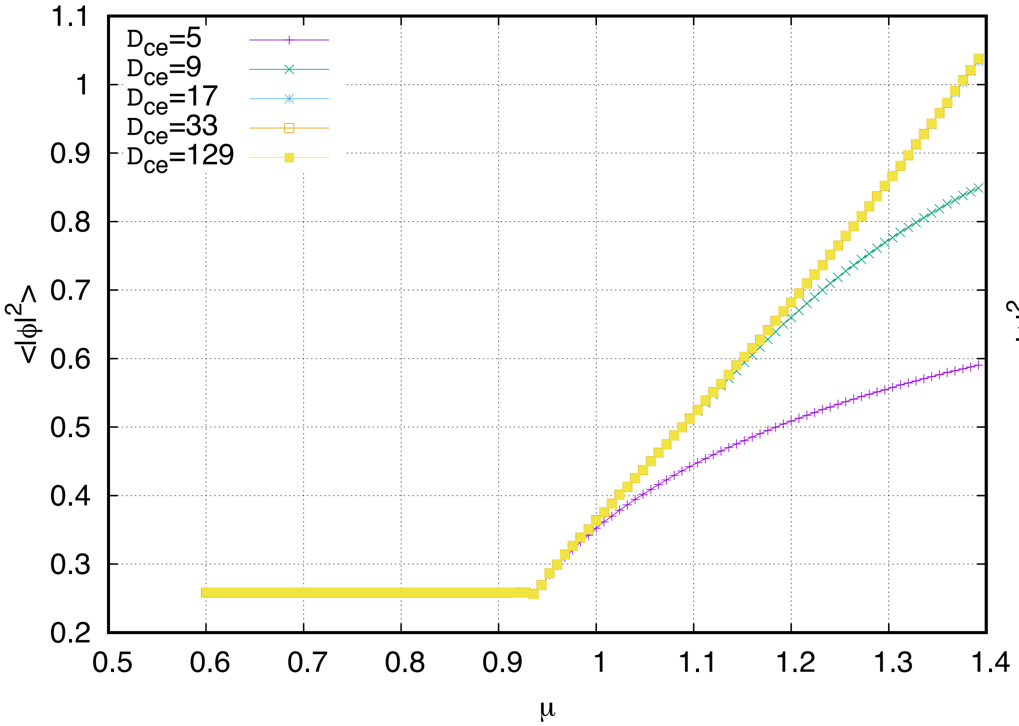
TRG should work for Z(positive)

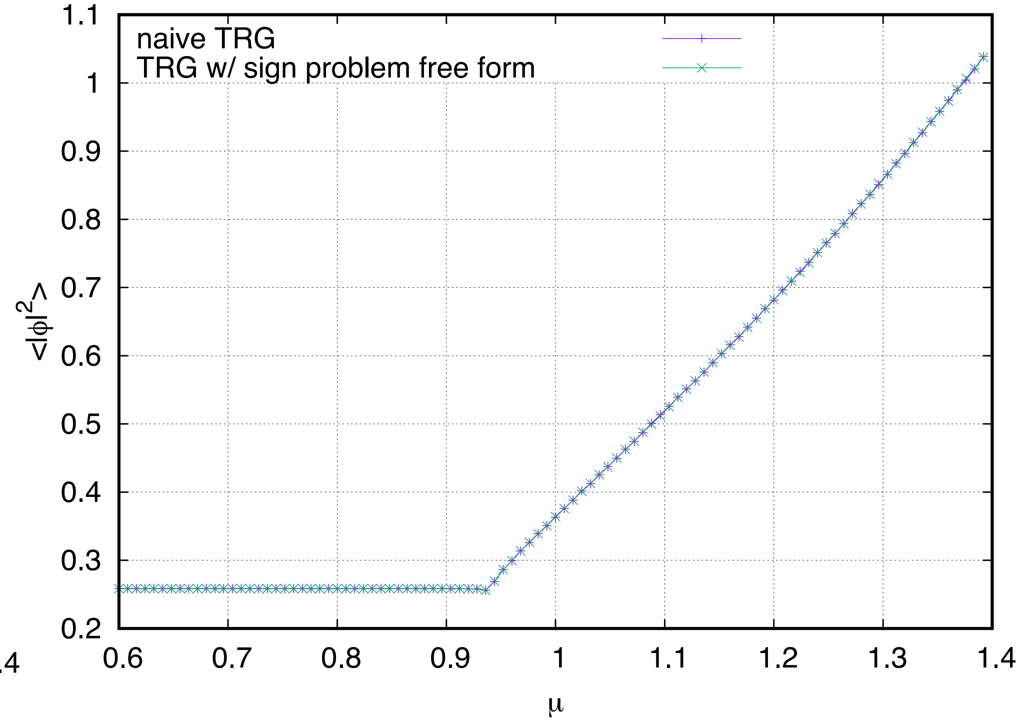
Consistency check btw the results for Z(original) and Z(positive)

Comparison btw Z(original) and Z(positive)

Kadoh+, in preparation

Parameters: m^2=0.01, V=1024x1024, λ=1, K=256, D_{cut}=64





Good convergence
in character expansion

Good agreement in large μ & V
⇒ Free from sign problem

Summary

What we have achieved so far

- Application of TRG/HOTRG to 2D scalar, fermion and gauge theories

— Construction of TN rep. for scalar field theories

— Development of Grassmann TRG/HOTRG for fermion systems

— Successful analyses of 2D models w/ complex actions

- Analyses of higher dimensional models

— Phase transition of 3D Z_2 gauge theory

— Phase transition of 4D Ising model

Next step

- Analysis of 3D, 4D models

— Nambu—Jona-Lasinio model at finite density

— Non-Abelian gauge theories

— Parallelization of ATRG (finished)