An Inverse Problem for a Non-Linear Wave Equation and Inverse Problems in General Relativity

Matti Lassas

in collaboration with

Yaroslav Kurylev,
Lauri Oksanen,
Gunther Uhlmann,
Yang Yang.

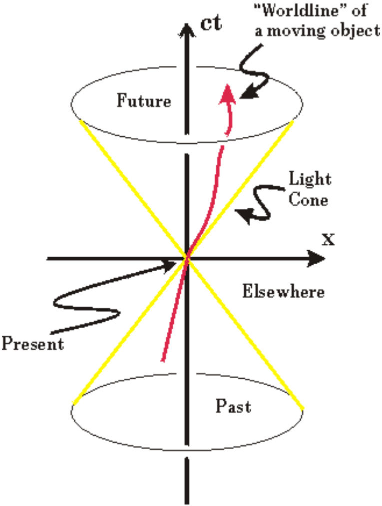
Finnish Centre of Excellence
in Inverse Problems Research

UNIVERSITY OF HELSINKI

Outline:

- Lorentzian manifolds and inverse travel time problems
- Passive measurements with point sources
- Inverse problem for non-linear wave equation
- Einstein-scalar field equations

Definitions



Let (M, g) be a Lorentzian manifold,

where the metric g is semi-definite.

$T_x M$ is the space of tangent vectors at x

$\xi \in T_x M$ is light-like if $g(\xi, \xi) = 0$, $\xi \neq 0$.

$\xi \in T_x M$ is time-like if $g(\xi, \xi) < 0$.

A curve $\mu(s)$ is time-like if $\dot{\mu}(s)$ is time-like.

Example: Minkowski space $\mathbb{R}^{1+3}$.

Coordinates $(x^0, x^1, x^2, x^3) \in \mathbb{R}^{1+3}$,

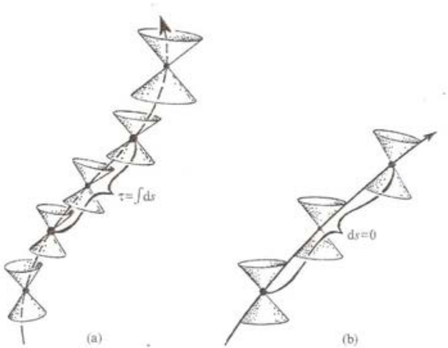
$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

The proper time is the elapsed time between two events as measured by a clock that passes through both events.

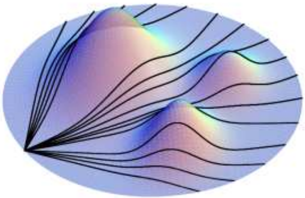
The proper time (Lorentzian length) along a causal curve $\mu([a, b])$ is

$$T(\mu) = \int_a^b \sqrt{-g(\dot{\mu}(s), \dot{\mu}(s))}\, ds.$$

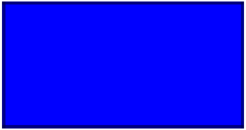
Causal geodesics maximize locally the Lorentzian length. Geodesics satisfy equation $\nabla_{\dot{\mu}(s)}\dot{\mu}(s) = 0$.
We denote by $\gamma_{x,\xi}(t)$ the geodesic with the initial point (x, ξ).



When distances of boundary points determine the metric?

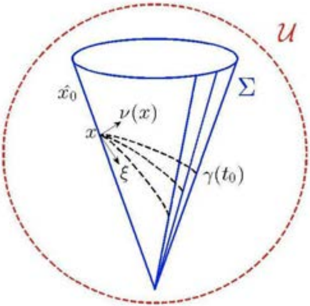


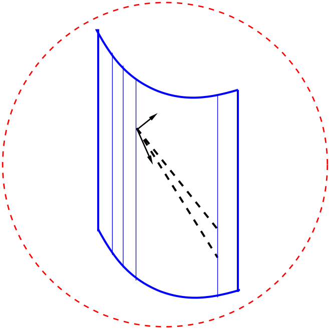
Riemannian manifolds:
Michel (1981),
Gromov (1983),
Croke (1990), Otal (1990),
L.- Sharafutdinov-Uhlmann (2003),
Pestov-Uhlmann (2005),
Stefanov-Uhlmann (2005),
Burago-Ivanov (2010),
Salo-Paternain-Uhlmann (2013)



Lorentzian manifolds:
Andersson-Dahl-Howard: Boundary and lens rigidity of Lorentzian surfaces. (TAMS 1996)

Figures: IMA





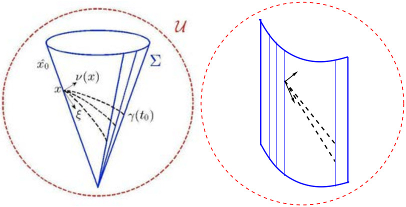
Definition: *A submanifold* $\Sigma \subset M$ *is time-like convex near a time-like vector* $(x_0, \xi_0) \in T\Sigma$ *if: For all* (x, ξ) *in a small neighborhood of* (x_0, ξ_0) *in* $T\Sigma$ *and* $r \in (0, r_0)$, *the geodesic* $\gamma(t) = \gamma_{x,\xi+r\nu}(t)$ *satisfies* $\gamma(t_0) \in \overline{\Sigma}$ *for some* $t_0 > 0$.

Theorem (M.L.-Oksanen-Yang)

Let (M_1, g_1) and (M_2, g_2) be Lorentzian manifolds, $\mathcal{U}_j \subset M_j$ be simply convex open sets, and $\Sigma_j \subset \mathcal{U}_j$ be time-like submanifolds that are time-like convex near $(x_j, \xi_j) \in T\Sigma_j$. Suppose that there is a diffeomorphism $\Phi : \Sigma_1 \to \Sigma_2$, $\Phi_*(x_1, \xi_1) = (x_2, \xi_2)$ and the Lorentzian distance functions satisfy

$$d_1(x, y) = d_2(\Phi(x), \Phi(y)), \qquad \text{for all } x, y \in \Sigma_1.$$

Then the derivatives $\partial^\alpha g_1$ at x_1 and $\partial^\alpha g_2$ at x_2 are the same.



When (M_1, g_1) is real-analytic, we can try to use analytic continuation to reconstruct it.

A function $F : M \to \mathbb{R}$ is a scalar curvature invariant if

$$F(x) = f(g(x), R(x), \nabla R(x), \ldots, \nabla^k R(x)), \quad k \in \mathbb{N},$$

where f is a smooth function, g is the metric tensor, R is the curvature tensor, and ∇ is the covariant differentiation.

Example: the Kretschmann scalar is $R_{abcd} R^{abcd}$.

Definition

(M, g) is geodesically complete modulo scalar curvature singularities if every maximal geodesic $\gamma : (\ell_-, \ell_+) \to M$ satisfies $\ell_\pm = \pm\infty$ or there is a scalar curvature invariant F such that $F(\gamma(t))$ is unbounded as $t \to \ell_\pm$.

Theorem (M.L.-Oksanen-Yang)

Let (M_1, g_1) and (M_2, g_2) be two smooth Lorentzian manifolds satisfying the assumptions of Theorem 1. Assume that (M_1, g_1) and (M_2, g_2) are connected, geodesically complete modulo scalar curvature singularities and real-analytic. Then the universal Lorentzian covering spaces of (M_1, g_1) and (M_2, g_2) are isometric.

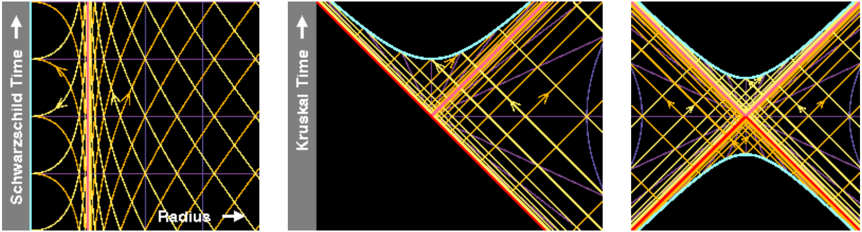
Schwarzschild black hole

The non-extended Schwarzschild black hole with Schwarzschild radius R_s in standard coordinates

$$(t, r, \theta, \phi) \in \mathbb{R} \times \mathbb{R}_+ \times (-\pi/2, \pi/2) \times (-\pi, \pi)$$

has the metric

$$g = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) .$$



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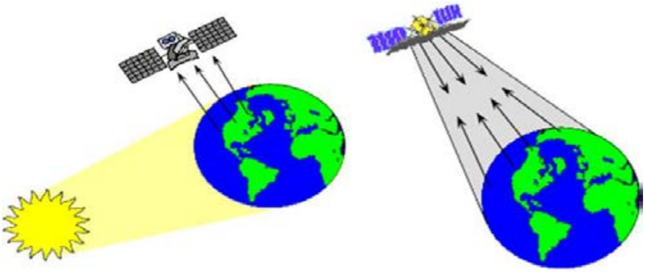
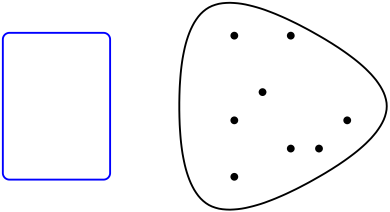


Passive sensors

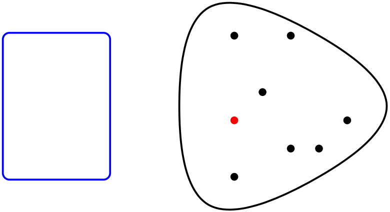
Active sensors

Inverse problem for passive measurements



Can one determine the metric when we observe wavefronts produced by point sources? - Yes in Riem. geometry (L.-Saksala)

Inverse problem for passive measurements



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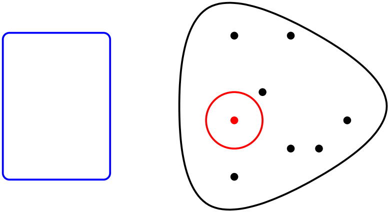

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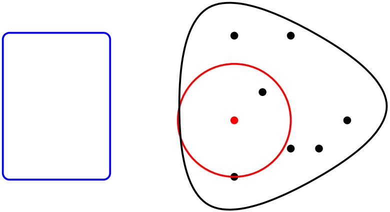

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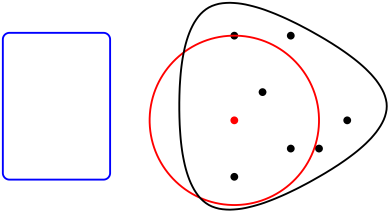

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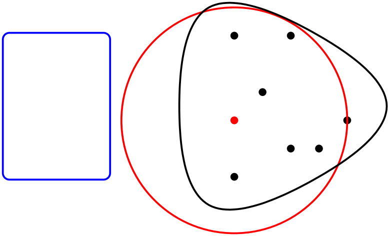

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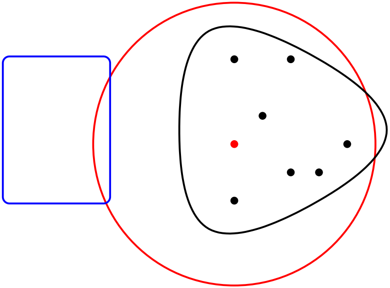

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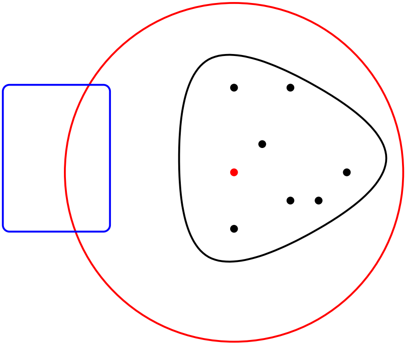

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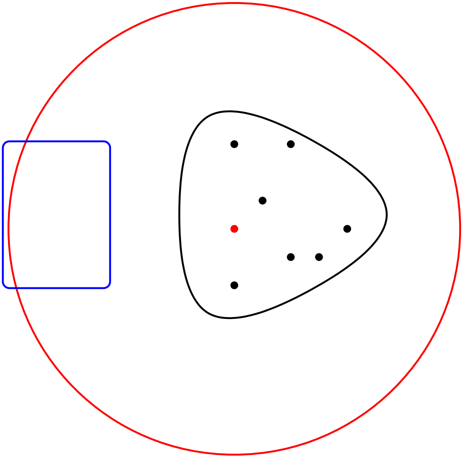

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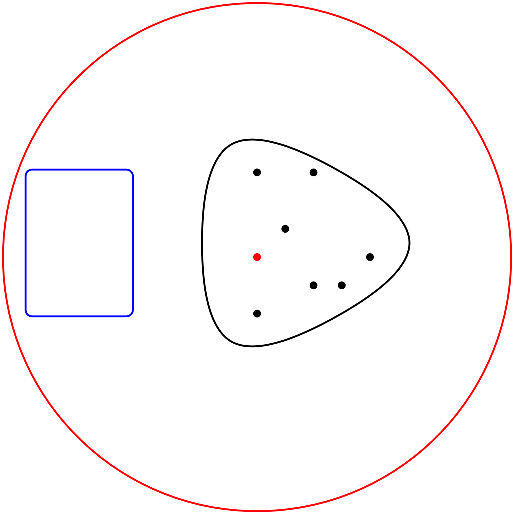

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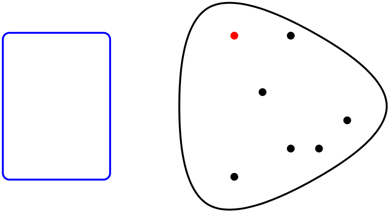

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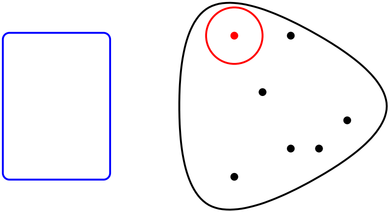

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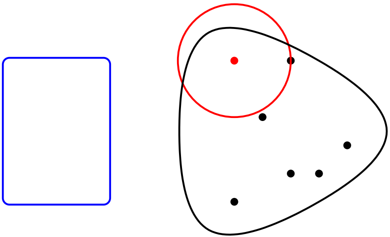

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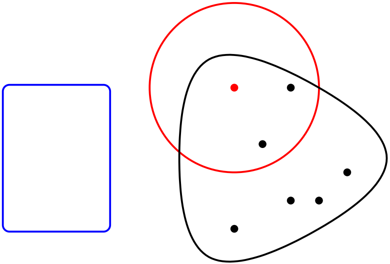

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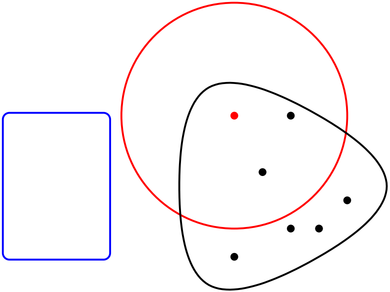

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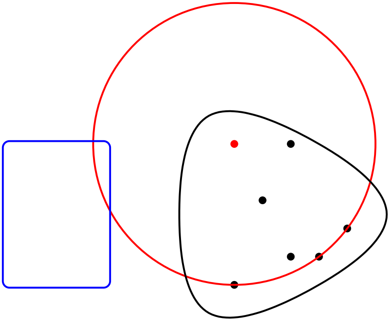

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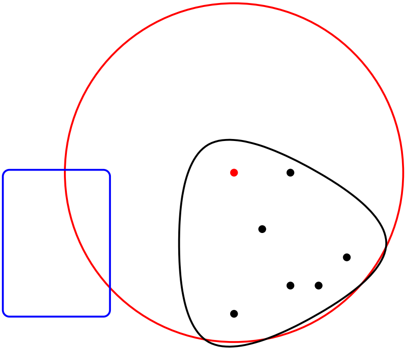

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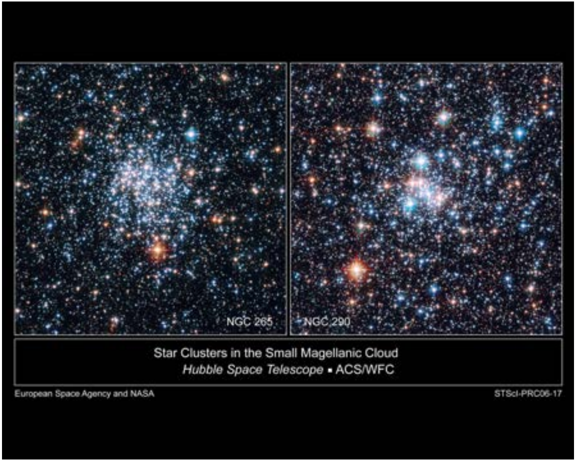
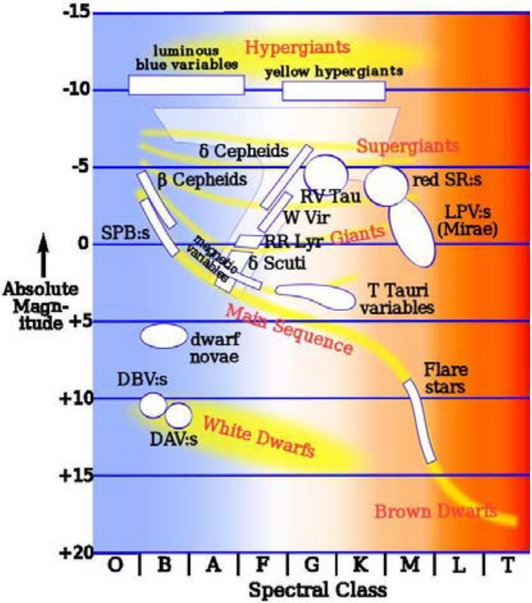
Inverse problems in space-time: Passive measurements

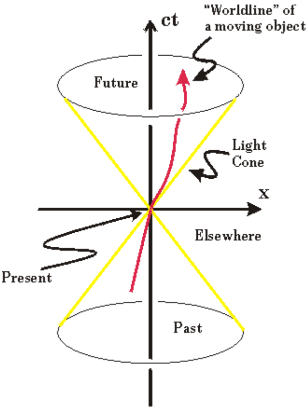


Can we determine structure of the space-time when we see light coming from many point sources that vary in time?



Variable stars in Hertzsprung-Russell diagram on star types.
Areas with variable stars are marked by white.

Definitions



Let (M, g) be a Lorentzian manifold.

$L_q M = \{\xi \in T_q M \setminus 0;\ g(\xi, \xi) = 0\}$,

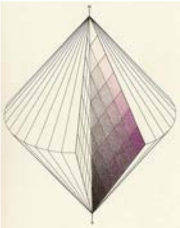
$L_q^+ M \subset L_q M$ is the future light cone,

$J^+(q) = \{x \in M;\ x \text{ is in causal future of } q\}$,

$J^-(q) = \{x \in M;\ x \text{ is in causal past of } q\}$,

$\gamma_{x,\xi}(t)$ is a geodesic with the initial point (x, ξ),

$\mathcal{L}_q^+ = \{\gamma_{q,\xi}(r) \in M;\ \xi \in L_q^+ M,\ r \geq 0\}$.



(M, g) is globally hyperbolic if

there are no closed causal curves and the set

$J^-(p_1) \cap J^+(p_2)$ is compact for all $p_1, p_2 \in M$.

Then M can be represented as $M = \mathbb{R} \times N$.

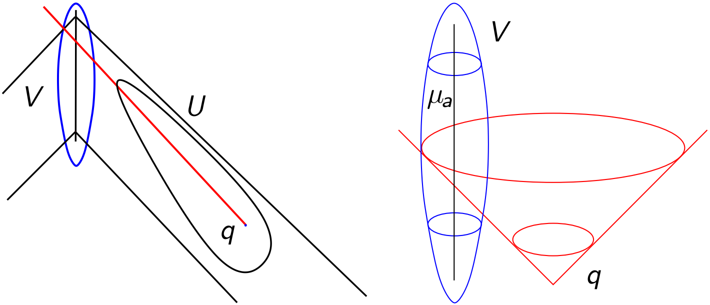
More definitions

Let $A \subset \mathbb{R}^m$ be open and $\mu_a : (-1,1) \to M$, $a \in A$ be a family of time-like geodesics such that $V = \bigcup_{a \in A} \mu_a(-1,1)$ is open.
We consider observations in V. Let $p^-, p^+ \in \mu_{a_0}$.

Let $U \subset J^-(p^+) \setminus J^-(p^-)$ be an open, relatively compact set.

The observation time function $F_q : A \to \mathbb{R}$ for a point $q \in U$ is

$$F_q(a) = \inf\{s \in \mathbb{R} \quad ; \quad \text{there is a future-directed light-like geodesic from } q \text{ to } \mu_a(s)\}$$

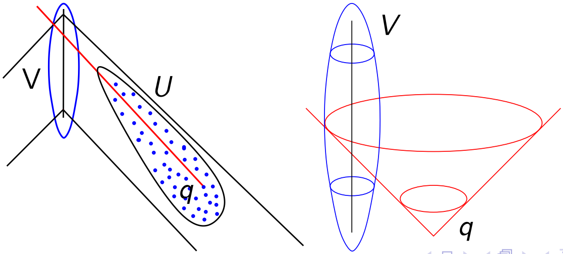


Theorem (Kurylev-L.-Uhlmann)

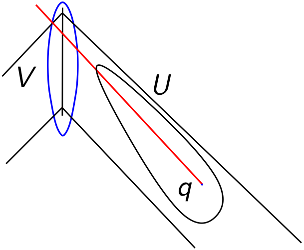
Let (M, g) be a globally hyperbolic Lorentzian manifold of dimension $n \geq 3$. Assume that $\mu_a(-1,1) \subset M$, $a \in A \subset \mathbb{R}^m$ are time-like geodesics, $V = \cup_{a\in A}\mu_a$ is open, and $p^-, p^+ \in \mu_{a_0}$. Let $U \subset J^-(p^+) \setminus J^-(p^-)$ be a relatively compact open set. Then $(V, g|_V)$ and the collection of the observation time functions,

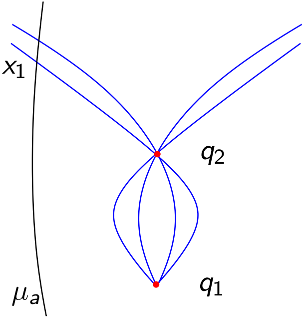
$$\mathcal{F}_U = \left\{ F_q : A \to \mathbb{R} \;\middle|\; q \in U \right\},$$

determine the set U, up to a change of coordinates, and the conformal class of the metric g in U.



Reconstruction of the topological structure of U

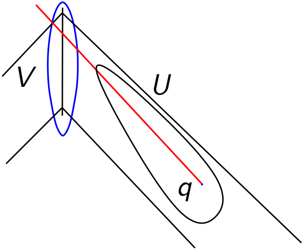


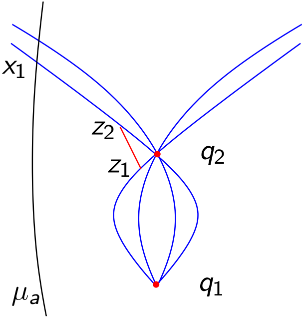


Assume that $q_1, q_2 \in U$ are such that $F_{q_1} = F_{q_2}$.
Then all light-like geodesics from q_1 to V go through q_2.

Let $x_1 = \mu_a(F_{q_1}(a))$.

Reconstruction of the topological structure of U





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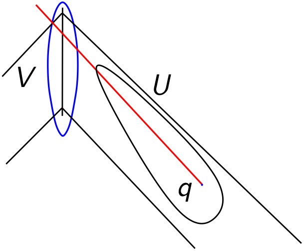
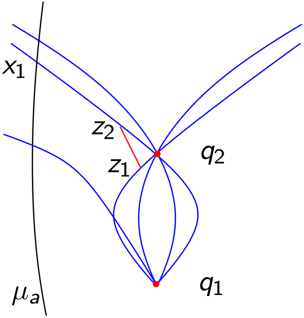
Then all light-like geodesics from q_1 to V go through q_2.

Let $x_1 = \mu_a(F_{q_1}(a))$.
Using a short cut argument we see that there is a causal curve from q_1 to x_1 that is not a geodesic.

Reconstruction of the topological structure of U





Assume that $q_1, q_2 \in U$ are such that $F_{q_1} = F_{q_2}$. Then all light-like geodesics from q_1 to V go through q_2.

Let $x_1 = \mu_a(F_{q_1}(a))$. Using a short cut argument we see that there is a causal curve from q_1 to x_1 that is not a geodesic.

This implies that q_1 can be observed on μ_a before x_1.

The map $\mathcal{F} : q \mapsto F_q$ is continuous and one-to-one. As $\overline{U}$ is compact, the map $\mathcal{F} : \overline{U} \to \mathcal{F}(\overline{U})$ is a homeomorphism.

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- Lorentzian manifolds and inverse travel time problems
- Passive measurements with point sources
- Inverse problem for non-linear wave equation

"Can we image a wave using other waves?"

- Einstein-scalar field equations

Next we consider inverse problems with active measurements.

We will consider inverse problems for non-linear wave equations, e.g.

$$\frac{\partial^2}{\partial t^2}u(t,y) - c(t,y)^2\Delta u(t,y) + a(t,y)u(t,y)^2 = f(t,y).$$

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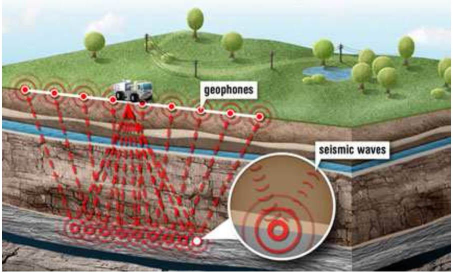
We will show that:

-Non-linearity helps to solve the inverse problem,

-"Scattering" from the interacting wave packets determines the structure of the spacetime.

Some results for hyperbolic inverse problems for linear equations:

- Belishev-Kurylev 1992 and Tataru 1995: Reconstruction of a Riemannian manifold with time-indepedent metric.
  The used unique continuation fails for non-real-analytic time-depending coefficients (Alinhac 1983).
- Eskin 2008: Wave equation with time-depending (real-analytic) lower order terms.
- Helin-L.-Oksanen 2012: Combining several measurements for together for the wave equation.



Non-linear wave equation in space-time

Let $M = \mathbb{R} \times N$, $\dim(M) = 4$. Consider the equation

$$\square_g u(x) + a(x)\, u(x)^2 = f(x) \quad \text{on } M_1 = (-\infty, T) \times N,$$
$$u(x) = 0 \quad \text{for } x = (x^0, x^1, x^2, x^3) \in (-\infty, 0) \times N,$$

where

$$\square_g u = \sum_{p,q=0}^{3} |\det(g(x))|^{-\frac{1}{2}} \frac{\partial}{\partial x^p} \left(|\det(g(x))|^{\frac{1}{2}} g^{pq}(x) \frac{\partial}{\partial x^q} u(x) \right)$$

and $a(x)$ is a non-vanishing C^∞-smooth function.

Alternative model:

$$\frac{\partial^2}{\partial t^2} u(t,y) - c(t,y)^2 \Delta u(t,y) + a(t,y) u(t,y)^2 = f(t,y), \quad x = (t,y).$$

Inverse problem for non-linear wave equation

Consider the equation

$$\begin{aligned} &\Box_g u(x) + a(x)\, u(x)^2 = f(x) \quad \text{on } M_1 = (-\infty, T) \times N, \\ &u(x) = 0 \quad \text{for } x \in (-\infty, 0) \times N, \end{aligned}$$

where the source $f \in C_0^6(V)$ is supported in an open set $V \subset M_1$.

In a neighborhood $\mathcal{W} \subset C_0^6(V)$ of the zero-function we define the measurement operator (source-to-solution operator),

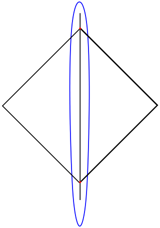
$$L_V : f \mapsto u|_V, \quad f \in \mathcal{W} \subset C_0^6(V).$$

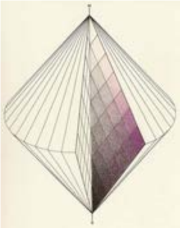
Theorem (Kurylev-L.-Uhlmann)

Let (M, g) be a globally hyperbolic Lorentzian manifold of dimension $(1+3)$. Let μ be a time-like path containing p^- and p^+, $V \subset M$ be a neighborhood of μ, and $a(x)$ be a nowhere vanishing function. Consider the non-linear wave equation

$$\begin{aligned} \square_g u(x) + a(x)\, u(x)^2 &= f(x) \quad \text{on } M_1 = (-\infty, T) \times N, \\ u &= 0 \quad \text{in } (-\infty, 0) \times N, \end{aligned}$$

where $supp(f) \subset V$. Then $(V, g|_V)$ and the measurement operator $L_V : f \mapsto u|_V$ determine the set $J^+(p^-) \cap J^-(p^+) \subset M$, up to a change of coordinates, and the conformal class of g in the set $J^+(p^-) \cap J^-(p^+)$.





Idea of the proof: Non-linear geometrical optics.

The non-linearity helps in solving the inverse problem.

Let $u = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \varepsilon^4 w_4 + E_\varepsilon$ satisfy

$$\Box_g u + au^2 = f, \quad \text{on } M_1 = (-\infty, T) \times N,$$
$$u|_{(-\infty,0)\times N} = 0$$

with $f = \varepsilon f_1$, $\varepsilon > 0$.
When $Q = \Box_g^{-1}$, we have

$$\begin{aligned} w_1 &= Qf_1, \\ w_2 &= -Q(a\, w_1\, w_1), \\ w_3 &= 2Q(a\, w_1\, Q(a\, w_1\, w_1)), \\ w_4 &= -Q(a\, Q(a\, w_1\, w_1)\, Q(a\, w_1\, w_1)) \\ &\quad -4Q(a\, w_1\, Q(a\, w_1\, Q(a\, w_1\, w_1))), \end{aligned}$$
$$\|E_\varepsilon\| \leq C\varepsilon^5.$$

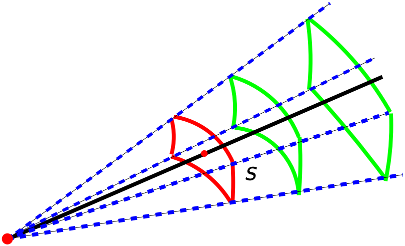
Interaction of waves in Minkowski space $\mathbb{R}^4$

Let x^j, $j = 1,2,3,4$ be coordinates such that

$$K_j = \{x^j = 0\}, \quad j = 1,2,3,4,$$

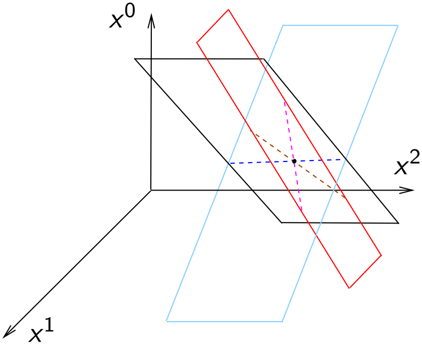
are light-like. We consider plane waves

$$u_j(x) = v \cdot (x^j)_+^m, \quad (s)_+^m = |s|^m H(s), \quad v \in \mathbb{R}, \ j = 1,2,3,4.$$



The interaction of the waves $u_j(x)$ produce new sources on

$$\begin{aligned} K_{12} &= K_1 \cap K_2, \\ K_{123} &= K_1 \cap K_2 \cap K_3 = \text{line}, \\ K_{1234} &= K_1 \cap K_2 \cap K_3 \cap K_3 = \{q\} = \text{one point}. \end{aligned}$$



Interaction of two waves

If we consider sources $f_{\vec{\varepsilon}}(x) = \varepsilon_1 f_1(x) + \varepsilon_2 f_2(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)$, and the corresponding solution $u_{\vec{\varepsilon}}$ of the wave equation, we have

$$W_2(x) = \frac{\partial}{\partial \varepsilon_1} \frac{\partial}{\partial \varepsilon_2} u_{\vec{\varepsilon}}(x)\Big|_{\vec{\varepsilon}=0} = \square_g^{-1}(a\, u_1 \cdot u_2),$$

where $u_j = \square_g^{-1} f_j$.

All light-like co-vectors in the normal bundle of $K_1 \cap K_2$ are in $N^* K_1 \cup N^* K_2$.
Thus no interesting singularities are produced by the interaction of two waves. (Greenleaf-Uhlmann '93)

Interaction of three waves

Consider sources

$$f_{\vec{\varepsilon}}(x) = \sum_{j=1}^{3} \varepsilon_j f_j(x), \quad \vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3),$$

and let $u_{\vec{\varepsilon}}$ be the solution with source $f_{\vec{\varepsilon}}$.
We have

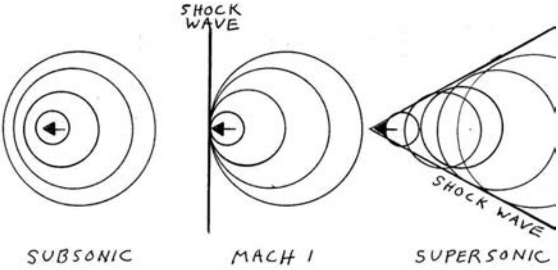
$$\begin{aligned} W_3 &= \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3} u_{\vec{\varepsilon}}\Big|_{\vec{\varepsilon}=0} \\ &= \square_g^{-1}(a\, u_1 \cdot \square_g^{-1}(a u_2 \cdot u_3)) + \dots \end{aligned}$$

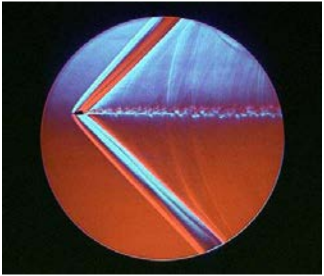
The interaction of the three waves happens on the line $K_{123} = K_1 \cap K_2 \cap K_2$ and produce new singularities.

Similar results in $\mathbb{R}^{1+2}$: Rauch-Reed '82 and Melrose-Ritter '85.
Examples with caustics: Joshi-Sa Barreto '98, Zworski '94.

Interaction of waves:

The non-linearity helps in solving the inverse problem.
Artificial sources can be created by interaction of waves using the non-linearity of the wave equation.





The interaction of 3 waves creates a point source in space that seems to move at a higher speed than light, that is, it appears like a tachyonic point source, and produces a new “shock wave” type singularity.

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Interaction of three waves.

Interaction of four waves

Consider sources $f_{\vec{\varepsilon}}(x) = \sum_{j=1}^{4} \varepsilon_j f_j(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, the corresponding solution $u_{\vec{\varepsilon}}$, and

$$W_4 \quad = \quad \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3}\partial_{\varepsilon_4} u_{\vec{\varepsilon}}(x)\Big|_{\vec{\varepsilon}=0}.$$

Since $K_{1234} = \{q\}$. Thus, when the four waves intersect, an artificial point source appears.

Interaction of four waves.

The 3-interaction produces conic waves (only one is shown below).

The 4-interaction produces a spherical wave from the point q that determines the observation times $F_q(a)$.

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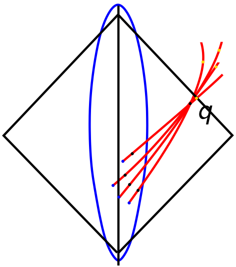
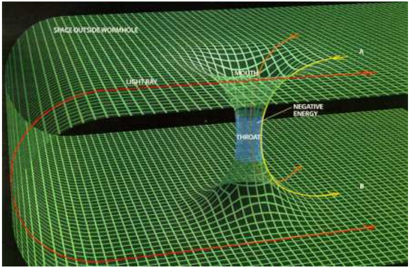
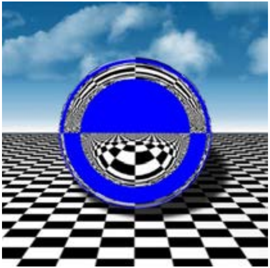


Outline:

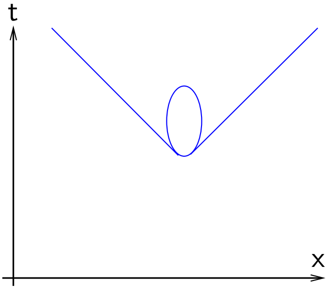
- Lorentzian manifolds and inverse travel time problems
- Passive measurements with point sources
- Inverse problem for non-linear wave equation
- Einstein-scalar field equations

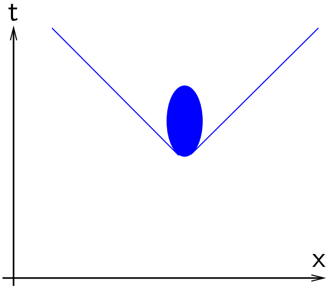




Figures: Anderson institute and Greenleaf-K.-L.-U.

Active measurements: two alternative events.





$$(\partial_t^2 - \partial_x^2)u(t,x) = 0,$$
$$u(0,x) = 0, \quad \partial_t u(0,x) = 0$$

$$(\partial_t^2 - \partial_x^2)u(t,x) = f(t,x),$$
$$u(0,x) = 0, \quad \partial_t u(0,x) = 0$$

In active measurements we consider a source that is either "on" or "off". Thus the physical model has to include free parameters that we can control.

To have a model where the total energy of the system is conserved, we need a model for the device that produces the source.

Example: seismic imaging with explosions

A model for the acoustic wave $u(x,t)$ and the pressure $p(x,t)$ in a source explosion, and the detonator $h(x,t)$ is

$$(\partial_t^2 - c(x)^2\Delta)u(x,t) = p(x,t),$$

Equations for $p(x,t)$ and $h(x,t)\ldots$

To build a model where the total energy of the system is conserved we need to model how the chemical energy of the explosive transforms to kinetic energy.

Einstein equations

The Einstein equation for the $(-,+,+,+)$-type Lorentzian metric g_{jk} of the space time is

$$\mathrm{Ein}_{jk}(g) = T_{jk},$$

where

$$\mathrm{Ein}_{jk}(g) = \mathrm{Ric}_{jk}(g) - \frac{1}{2}(g^{pq}\,\mathrm{Ric}_{pq}(g))g_{jk}.$$

In vacuum, $T = 0$. In wave map coordinates, the Einstein equation yields a quasilinear hyperbolic equation and a conservation law,

$$g^{pq}(x)\frac{\partial^2}{\partial x^p \partial x^q} g_{jk}(x) + B_{jk}(g(x), \partial g(x)) = T_{jk}(x),$$
$$\nabla_p(g^{pj}\,T_{jk}) = 0.$$

One can not do measurements in vacuum, so matter fields need to be added. We can consider the coupled Einstein and scalar field equations with sources,

$$\begin{aligned} &\mathrm{Ein}(g) = T, \quad T = \mathbf{T}(\phi, g) + \mathcal{F}_1, \quad \text{on } (-\infty, t_0) \times N, \\ &\square_g \phi_\ell - m^2 \phi_\ell = \mathcal{F}_2^\ell, \quad \ell = 1, 2, \dots, L, \\ &g|_{t<0} = \widehat{g}, \quad \phi|_{t<0} = \widehat{\phi}. \end{aligned} \tag{1}$$

Here, $\widehat{g}$ and $\widehat{\phi}$ are C^∞-smooth background solutions that satisfy equations (1) with the zero sources and

$$\mathbf{T}_{jk}(g, \phi) = \sum_{\ell=1}^{L} \partial_j \phi_\ell \, \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_\ell \, \partial_q \phi_\ell - \frac{1}{2} m^2 \phi_\ell^2 g_{jk}.$$

To obtain a physically meaningful model, the stress-energy tensor T needs to satisfy the conservation law

$$\nabla_p (g^{pj} T_{jk}) = 0, \quad k = 1, 2, 3, 4.$$

Assume that $(g_\varepsilon, \phi_\varepsilon)$ depend smoothly on $\varepsilon \in [0, \varepsilon_0)$ and solve the non-linear Einstein equations with sources $(\mathcal{F}^1_\varepsilon, \mathcal{F}^2_\varepsilon)$ and the conservation law

$$\nabla^{g_\varepsilon}_j(\mathbf{T}^{jk}(g_\varepsilon, \phi_\varepsilon) + (\mathcal{F}^1_\varepsilon)^{jk}) = 0, \quad k = 1, 2, 3, 4.$$

Assume that $g_\varepsilon|_{\varepsilon=0} = \widehat{g}$, $\phi_\varepsilon|_{\varepsilon=0} = \widehat{\phi}$, and $\mathcal{F}_\varepsilon|_{\varepsilon=0} = 0$.

Then $\dot{g} = \partial_\varepsilon g_| e|_{\varepsilon=0}$, $\dot{\phi} = \partial_\varepsilon \phi_\varepsilon|_{\varepsilon=0}$ and $f^j = \partial_\varepsilon \mathcal{F}^1_\varepsilon|_{\varepsilon=0}$, satisfy the linearized conservation law

$$\sum_{\ell=1}^L f^2_\ell\, \partial_j \widehat{\phi}_\ell + \tfrac{1}{2} \widehat{g}^{pk} \widehat{\nabla}_p f^1_{kj} = 0, \quad j = 1, 2, 3, 4.$$

Definition

Linearization stability (Choquet-Bruhat, Deser, Fischer, Marsden)
Let $f = (f^1, f^2)$ satisfy the linearized conservation law

$$\sum_{\ell=1}^{L} f_\ell^2\, \partial_j \widehat{\phi}_\ell + \frac{1}{2}\widehat{g}^{pk}\widehat{\nabla}_p f_{kj}^1 = 0, \quad j = 1,2,3,4 \tag{2}$$

and let $(\dot{g}, \dot{\phi})$ be the corresponding solution of the linearized Einstein equation. We say that f has the Linearization Stability (LS) property if there is $\varepsilon_0 > 0$ and families

$$\begin{aligned}
\mathcal{F}_\varepsilon &= (\mathcal{F}_\varepsilon^1, \mathcal{F}_\varepsilon^2) = \varepsilon f + O(\varepsilon^2), \\
g_\varepsilon &= \widehat{g} + \varepsilon \dot{g} + O(\varepsilon^2), \\
\phi_\varepsilon &= \widehat{\phi} + \varepsilon \dot{\phi} + O(\varepsilon^2),
\end{aligned}$$

where $\varepsilon \in [0, \varepsilon_0)$, such that $(g_\varepsilon, \phi_\varepsilon)$ solves the non-linear Einstein equations and the conservation law

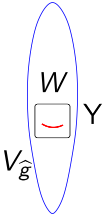
$$\nabla_j^{g_\varepsilon}(\mathbf{T}^{jk}(g_\varepsilon, \phi_\varepsilon) + (\mathcal{F}_\varepsilon^1)^{jk}) = 0, \quad k = 1,2,3,4.$$

Let $V_{\widehat{g}} \subset M$ be a open set that is a union of freely falling geodesics that are near μ, $L \geq 4$.

Condition A: Assume that at any $x \in V_{\widehat{g}}$ the 4×4 matrix

$$A(x) = \Big[\big(\partial_j \widehat{\phi}_\ell(x) \big)_{\ell,j=1}^4 \Big]$$

is invertible.



Theorem (Kurylev-L.-Oksanen-Uhlmann)

Let Condition A be valid and $W \subset V_{\widehat{g}}$ be open. Assume that $f = (f^1, f^2)$ satisfies the linearized conservation law and f is supported in W. Then f has a linearization stability property with a family $\mathcal{F}_\varepsilon$ supported in W.

An alternative formulation

We can also formulate the direct problem for the Einstein-scalar field equations. Let g and $\phi = (\phi_\ell)_{\ell=1}^L$ satisfy

$$\begin{aligned}
&\mathrm{Ein}_{jk}(g) = P_{jk} + \mathbf{T}_{jk}(g,\phi), \quad \text{on } (-\infty, t_0) \times N, \\
&\square_g \phi_\ell - m^2 \phi_\ell = S_\ell, \quad \ell = 1,2,3,\dots,L, \\
&S_\ell = Q_\ell + \mathcal{S}_\ell^{2nd}(g, \phi, \nabla\phi, Q, \nabla^g Q, P, \nabla^g P), \\
&g|_{t<0} = \widehat{g}, \quad \phi|_{t<0} = \widehat{\phi}.
\end{aligned}$$

Here Q and P_{jk} are considered as the primary sources.
The functions $\mathcal{S}_\ell^{2nd}$ need to be constructed so that the conservation law is satisfied for all solutions (g,ϕ). These functions correspond to a model for a measurement device.
When Condition A is satisfied, secondary source functions $\mathcal{S}_\ell^{2nd}$ can be constructed, for small Q and P, by solving a pointwise system of linear equations.

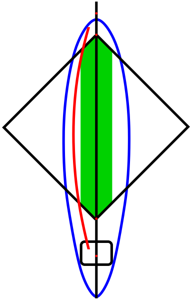
Let $V_{\widehat{g}} \subset M$ be a neighborhood of the geodesic μ and $p^-, p^+ \in \mu$.

Theorem (Kurylev-L.-Uhlmann)

Assume that the condition A is valid. Let $\varepsilon > 0$ be small and

$$\begin{aligned}\mathcal{D} \;=\; & \{(V_g, g|_{V_g}, \phi|_{V_g}, \mathcal{F}|_{V_g});\ g \text{ and } \phi \text{ satisfy Einstein equations} \\ & \quad \text{with a source } \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2),\ \text{supp}\,(\mathcal{F}) \subset V_g,\ \|\mathcal{F}\|_{C^6} < \varepsilon, \\ & \quad \nabla_j(\mathbf{T}^{jk}(g,\phi) + \mathcal{F}_1^{jk}) = 0\}.\end{aligned}$$

The data set $\mathcal{D}$ determines uniquely the conformal type of the double cone $(J^+(p^-) \cap J^-(p^+), \widehat{g})$.



Thank you for your attention!