On Riemannian Geometry of the Einstein Equation

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The Einstein-Maxwell System

Spacetime is a Lorentzian 4-manifold $(N^4, \mathbf{g})$ satisfying the Einstein equations

$$R_{ab} - \frac{1}{2} R\, \mathbf{g}_{ab} = T_{ab}$$

T is the energy-momentum-stress tensor of the matter fields. Here we assume it is determined by the electromagnetic field strength

$$T_{ab} = -\left(F_{ac} F_b^c + \frac{1}{4} F_{cd} F^{cd} \mathbf{g}_{ab} \right)$$

where F satisfies the Maxwell equations

$$\operatorname{div} F = \mathcal{J}, \qquad dF = 0$$

and $\mathcal{J}$ is the charge-current density.

The Einstein-Maxwell System

The Einstein equation $R_{ab} - \frac{1}{2} R\, \mathbf{g}_{ab} = T_{ab}$ is the Euler-Lagrange equation for the Hilbert-Einstein functional;

$$\mathcal{H}(\mathbf{g}) = \int_N \{R_{\mathbf{g}} + L\}\, d\mu_{\mathbf{g}}$$

where $R_{\mathbf{g}}$: the scalar curvature, R_{ab}: the Ricci curvature of the Lorentzian metric $\mathbf{g}$, and L: the Lagrangian for non-gravitational fields.

Vacuum = Ricci-flat

Recall the following rigidity result from Riemannian/Lorentzian geometry;

$$R_{abcd} = 0 \Rightarrow \text{ the space is flat; e.g. } \mathbb{R}^n, \mathbb{R}^{3,1}$$

When $L = 0$ $(\Leftrightarrow T_{ab} = 0)$, the vacuum Einstein equation (VEE) $R_{ab} - \frac{1}{2}R\,\mathbf{g}_{ab} = 0$ implies $R_{\mathbf{g}} = 0$, and hence

$$R_{ab} = 0;$$

the second simplest curvature equation conceivable.

Special Solutions I

The **simplest** solution to the VEE is given by

$$\mathbf{g} = -dt^2 + \delta, \qquad N^4 = \mathbb{R} \times \mathbb{R}^3$$

where δ is the standard metric on $\mathbb{R}^3$.

The **second simplest** solution to the VEE is the Schwarzschild solution, whose exterior region is given by

$$\mathbf{g} = -v^2 dt^2 + u^4 \delta, \qquad N^4 = \mathbb{R} \times \left(\mathbb{R}^3 \setminus B\right)$$

where $B = B_{m/2}(0)$ and

$$v = \frac{1 - m/2r}{1 + m/2r}, \qquad u = 1 + \frac{m}{2r}$$

The Riemannian metric $u^4 \delta$ is scalar flat.

How does a blackhole look like?

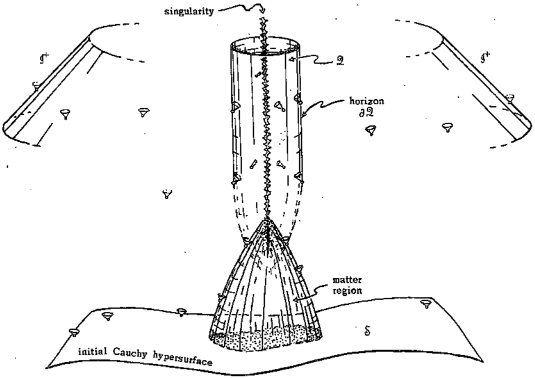

Figure 4. Dynamical collapse to a black hole.

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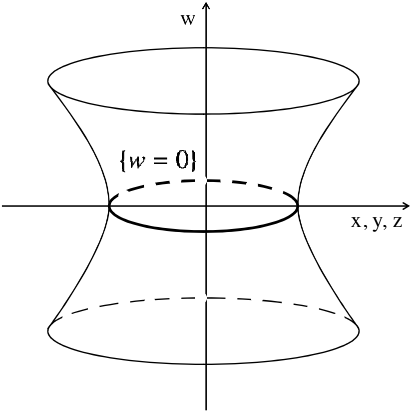


$\{t = \text{constant}\}$ Schwarzschild slice as a hypersurface in $\mathbb{R}^4$.

Special Solutions II

The exterior region of the third simplest solution; the Reissner-Nordström (1918) spacetime is given by

$$\mathbf{g} = -v^2 dt^2 + u^4 \delta, \qquad N^4 = \mathbb{R} \times \left(\mathbb{R}^3 \setminus B\right)$$

where $B = B_{\sqrt{m^2 - q^2}/2}(0)$ and

$$v = \frac{1 - (m^2 - q^2)/4r^2}{1 + m/r + (m^2 - q^2)/4r^2}, \qquad u = \sqrt{1 + \frac{m}{r} + \frac{m^2 - q^2}{4r^2}}$$

with electric and magnetic fields

$$E = u^{-6} \nabla \left(\frac{q}{r}\right), \qquad B \equiv 0, \qquad R(u^4 \delta) = 2|E|^2$$

$m \geq |q| \Rightarrow$ naked singularity. $q = 0$ case : Schwarzschild.

Special Solutions III

The Majumdar-Papapetrou (1947) spacetime is given by

$$\mathbf{g} = -u^2 dt^2 + u^{-2}\delta \qquad N^4 = \mathbb{R} \times \left(\mathbb{R}^3 \setminus \cup_{i=1}^{N}\{p_i\}\right)$$

where

$$u = \left(1 + \sum_{i=1}^{N} m_i/r_i\right)^{-1}, \qquad E = \nabla \log u, \qquad B \equiv 0.$$

$m_i > 0$ is both the mass and charge of each black hole, and r_i is the Euclidean distance to the point p_i.
We also have $R(u^{-2}\delta) = 2|E|^2$.
When $N = 1$, this reduces to the extremal Reissner-Nordström solution $m = |q|$.

Static Solution and Elliptic System

Theorem

(Chrusciel, Heusler, Bunting, Masood-ul-Alam, Tod) The only static black hole solutions of the Einstein-Maxwell system are the Reissner-Nordström and Majumdar-Papapetrou spacetimes.

Static Solution and Elliptic System

All the static Einstein-Maxwell solutions above can be written as

$$ds^2 = -V^2 dt^2 + g, \;\; A = \phi dt$$

together with

$$\begin{aligned} R_{ij} &= V^{-1}\nabla_i \nabla_j V - 2V^2 \nabla_a \phi \nabla_b \phi + V^{-2} |\nabla \phi|^2 g_{ij} \\ \triangle_g V &= V^{-1} |\phi|^2 \\ \triangle_g \phi &= V^{-1} \nabla_a \nabla^a \phi \end{aligned}$$

satisfied on the space-like hypersurface $\Sigma := \{t = \text{ const. }\}$ where $i = 1, 2, 3$. $V > 0$ satisfies the boundary conditions $V \to 1$ as $x \to \infty$ and $V|_{\partial\Sigma} = 0$.

Initial Data and Constraints

Theorem (Fundamental Theorem of Surface Theory (Bonnet))

Given an embedding data set (M, g, k) satisfying

$$\begin{cases} R_{ijkl} = k_{ij}k_{kl} - k_{il}k_{jk} & (\textit{Gauss equation}) \\ D_i k_{jk} - D_j k_{ik} = 0 & (\textit{Codazzi equation}) \end{cases}$$

there exists a unique (up to geometric motions) embedding $\iota : M^n \hookrightarrow \mathbb{R}^{n+1}$ so that g and k are the first and the second fundamental forms of ι.

Initial Data and Constraints for VEE

(M, g, k) is an initial data set for the Einstein system if the following set of constraint equations hold

$$\mu := 0 = \frac{1}{2}(R^{(3)} + (\operatorname{Tr} k)^2 - |k|^2)$$
$$J := 0 = \operatorname{div}(k - (\operatorname{Tr} k)g).$$

where μ and J are the energy and momentum density of the matter fields. (cf. $\mu = 0$:Tr^2(Gauss eqn), $J = 0$:Tr(Codazzi eqn))
The dominant energy condition is:

$$\mu \geq |J|$$

always satisfied when vacuum ($\mu = 0$ & $J = 0$).

Initial Data and Constraints for EME

(M, g, k, E, B) is an initial data set for the Einstein-Maxwell system if the following set of constraint equations hold

$$\mu := T(N, N) = \frac{1}{2}(R^{(3)} + (\mathrm{Tr}\, k)^2 - |k|^2)$$
$$J := T(N, \cdot) = \mathrm{div}(k - (\mathrm{Tr}\, k)g)$$
$$\mathrm{div}\, E = 0, \qquad \mathrm{div}\, B = 0.$$

where μ and J are the energy and momentum density of the matter fields.
The dominant energy condition is:

$$\mu \geq |J + 2E \times B| + |E|^2 + |B|^2$$

Initial Data and Constraints

Theorem (Choquet-Bruhat–Geroch)

Suppose that (M^3, g, k) is an initial data set for the Einstein equation. Then there exists a unique maximal development $(N^4, \mathbf{g})$ where (M^3, g) is isometrially embedded ($\mathbf{g}|_M = g$) with its exterior curvature k.

Corollary

Suppose that (M, g, k, E, B) is an initial data set for the Einstein-Maxwell equation. Then there exists a unique maximal development $(N^4, \mathbf{g}, F)$ where (M^3, g) is isometrically embedded ($\mathbf{g}|_M = g$) with its exterior curvature k and the electro-magnetic tensor F inducing E and B.

Asymptotic Flatness

Here we are concerned with time-symmetric data $k = 0$, and the purely electric case $B \equiv 0$. Then the constraints and the dominant energy condition are reduced to

$$R^{(3)} \geq 2|E|^2.$$

A time-symmetric ($k \equiv 0$) initial data set (M, g, E) is asymptotically flat if there is a compact set K such that $M \setminus K$ is the disjoint union of finitely many ends diffeomorphic to $\mathbb{R}^3 \setminus B$, and the fields have the following decay along those ends:

$$g_{ij} - \delta_{ij} = O_1(r^{-1}), \qquad E = O(r^{-2}).$$

Asymptotic Invariants

AF-condition $\Rightarrow SO(3,1)$ action on the sphere at the infinity $\Rightarrow$ preserved quantities by Noether's Theorem. (Hamiltonian Formulation)
Define the ADM mass and total charges by

$$m = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} (g_{ij,i} - g_{ii,j})\nu^j$$

$$q_e = \frac{1}{4\pi} \lim_{r\to\infty} \int_{S_r} E \cdot \nu, \qquad q_b = \frac{1}{4\pi} \lim_{r\to\infty} \int_{S_r} B \cdot \nu.$$

Newton versus Einstein

The Newtonian equation of motion is

$$\frac{d^2x}{dt^2} = -\nabla\phi, \qquad \triangle\phi = 4\pi\rho$$

where the motion is governed by the first derivative of a potential which is a solution of a second order elliptic PDE.
The Einstein equation is

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}, \qquad R_{ab} - \frac{1}{2}R\mathbf{g}_{ab} = T_{ab}$$

where the motion (geodesic equation) is governed by the first derivative (Γ^i_{jk}) of a potential $\mathbf{g}$, which is a solution of a system of second order hyperbolic PDE's.

Newton versus Einstein (continued)

The potential for the Newtonian equation of motion is of the form

$$\phi(x) = \frac{m}{r} + o(r^{-1})$$

where m is the total mass $\int \rho d\mu$
The potential for the equation of motion for the general relativity is

$$\mathbf{g}_{ab}|_{\{t=\text{const.}\}} = (1 + \frac{m}{2r} + o(r^{-1}))^4 \, \delta_{ab}$$

where m is the ADM mass for the asymptotically flat isolated system.

Positive Mass Theorems

In pure vacuum $E = B = 0$ (time-symmetry), the dominant energy condition reduces to $R^{(3)} \geq |k|^2$ ($R \geq 0$ resp.).

Theorem (Schoen–Yau 1979-81, Witten 1981)

Let (M, g, k) be an asymptotically flat initial data set satisfying the dominant energy condition, then $m \geq 0$ with equality if and only if $M \hookrightarrow$ Minkowski space.

Theorem (Gibbons, Hawking, Horowitz, Perry (1983))

Let (M, g, k, E, B) be an asymptotically flat initial data set satisfying the dominant energy condition, then $m \geq \sqrt{q_e^2 + q_b^2}$, and equality holds iff $(M, g, k, E, B) \hookrightarrow MP^4$.

Note that the case of equality was only established in (2006) by Chrusciel, Reall, Tod, assuming the maximal slice condition. The general case of equality is still open.

We now restrict to the time symmetric ($k = 0$) case. The black hole horizon S is the outermost minimal surface. Each component of such a surface must have spherical topology, and the topology of the exterior is trivial. We define the area radius as $r = \sqrt{A/4\pi}$, where A is the area of S.

Theorem (Bray, Huisken/Ilmanen, 2001)

Let (M, g) be an asymptotically flat Riemannian 3-manifold with an outermost minimal surface boundary of area radius r, and $R^3 \geq 0$. Then

$$m \geq \frac{r}{2},$$

with equality iff $(M, g) \cong \mathrm{Sch}^3$.

A Charged Penrose Inequality

A natural generalization of the Penrose inequality was proposed by Jang (1979), and later by Gibbons. In the pure electric ($B \equiv 0$), time-symmetric case, the dominant energy condition is $R^3 \geq 2|E|^2$.

Theorem (Jang, Huisken/Ilmanen)

Let (M, g, E) be an asymptotically flat time symmetric initial data set with a single component black hole boundary of mass m, area radius r, and charge q. If the dominant energy condition is satisfied, then

$$m \geq \frac{1}{2}\left(r + \frac{q^2}{r}\right)$$

In the case of equality, $(M, g, E) \cong RN^3$ (Khuri-Disconzi.)
What about the multiple black hole case?

Counterexample

The cross section of the neck in the two-neck Majumdar-Papapetrou solution MP_2 satisfies

$$m - \frac{1}{2}\left(r + \frac{q^2}{r}\right) = -\frac{1}{2r}(m-r)^2 < 0.$$

This suggests that the MP spacetime violates in spirit (MP is NOT AF!) "the charged Penrose inequality" unless there is only one black hole (MP_1).

Theorem (Weinstein–Yamada, 2005)

There exists an asymptotically flat initial data set (M, g, E) with a multiple component black hole boundary, and satisfying the dominant energy condition, for which the charged Penrose inequality is violated.

Physical Arguments

We note that the inequality $m \geq 1/2(r + q^2/r)$ is equivalent to:

$$m - \sqrt{m^2 - q^2} \leq r \leq m + \sqrt{m^2 - q^2}.$$

Penrose's heuristic physical arguments, based on cosmic censorship, yield only the upper inequality.
The counterexample above does not yield a violation of cosmic censorship:

$$4\pi r^2 = A = 4\pi \sum q_i^2 \leq 4\pi \left(\sum q_i\right)^2 = 4\pi q^2 \leq 4\pi m^2$$

in the Majumdar-Papapetrou spacetime. (Recall $|q| \leq m$.)

Main Theorem (arXiv:1409.3271)

Theorem (Gilbert Weinstein–Marcus Khuri–S.Y.)

Let (M, g, E) be an AF time-symmetric initial data set satisfying the dominant energy condition $R \geq 2|E|^2$, and having mass m, area radius r, and charge q. Then $r \leq m + \sqrt{m^2 - q^2}$ with equality if and only if the data is RN.

An Observation

We observe that the inequality

$$r \leq m + \sqrt{m^2 - q^2}$$

will follow from the following two inequalities:

$$m \geq |q| \qquad \text{if } r \leq q$$
$$m \geq \frac{1}{2}\left(r + \frac{q^2}{r}\right) \qquad \text{if } q < r$$

Note that $m \geq |q|$, i.e., the positive mass theorem with charge has been established, regardless of the relation between q and r. Hence it remains to establish the second inequality, assuming $q < r$.

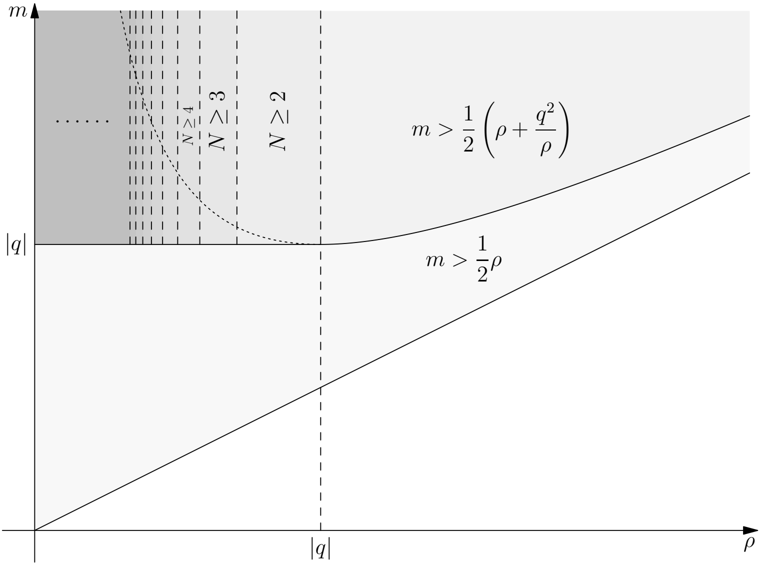
Juxtaposing the inequalities

We have the following set of inequalities.

$$\begin{aligned} m &\geq 0 \\ m &\geq \frac{1}{2}r \\ m &\geq \frac{1}{2}\left(r + \frac{q^2}{r}\right) \quad \text{if } r \geq |q| \\ m &\geq |q| \qquad \text{if } r < |q| \end{aligned}$$

Note that there is no topological restrictions to the inequalities and that the equality cases are given by the exact solutions.

Graphical representation of geometric inequalities



ADM formulation as an Inverse Problem

In summary,
The time-invariant Hamiltonians (the ADM mass m and the total charge q) are the *scattering data* observable at the infinity, and the Penrose type inequalities give a glimpse to the geometry of the finite regions, in particular the size of the blackhole (area radius ρ), which in turn tells the size of the source of gravitational force.