2022/9/27
TGSW 2022

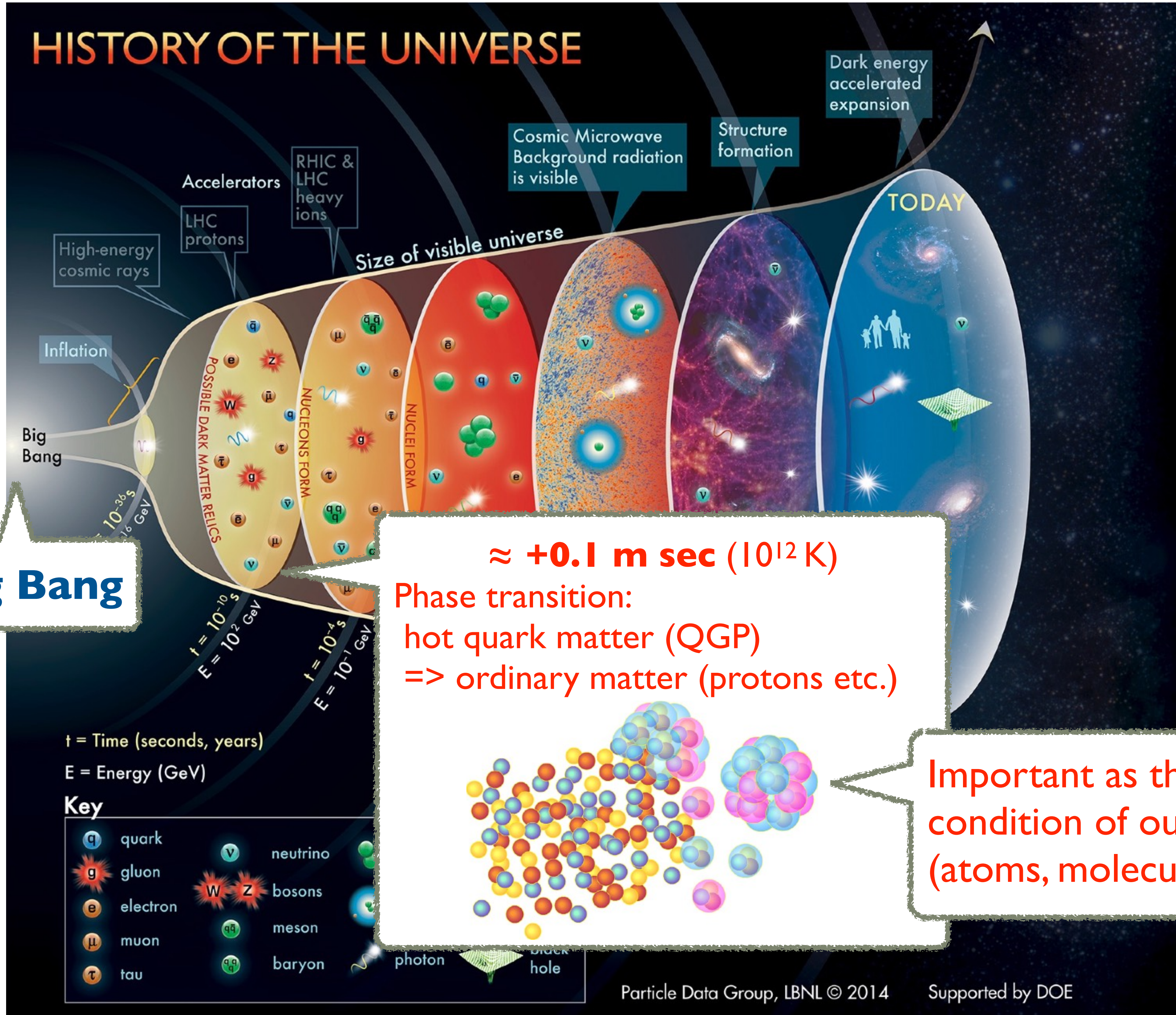
Critical point in heavy-quark QCD at finite temperature

Kazuyuki Kanaya[1) with **Ryo Ashikawa**[2), **Shinji Ejiri**[3),
Masakiyo Kitazawa[2,4), **Hiroshi Suzuki**[5), **Naoki Wakabayashi**[3)
(WHOT-QCD Collaboration)
1) Univ. Tsukuba, 2) Osaka Univ., 3) Niigata Univ., 4) Kyoto Univ., 5) Kyushu Univ.

University of Tsukuba

Tomonaga Center for the History of the Universe

QCD : the fundamental theory of quarks



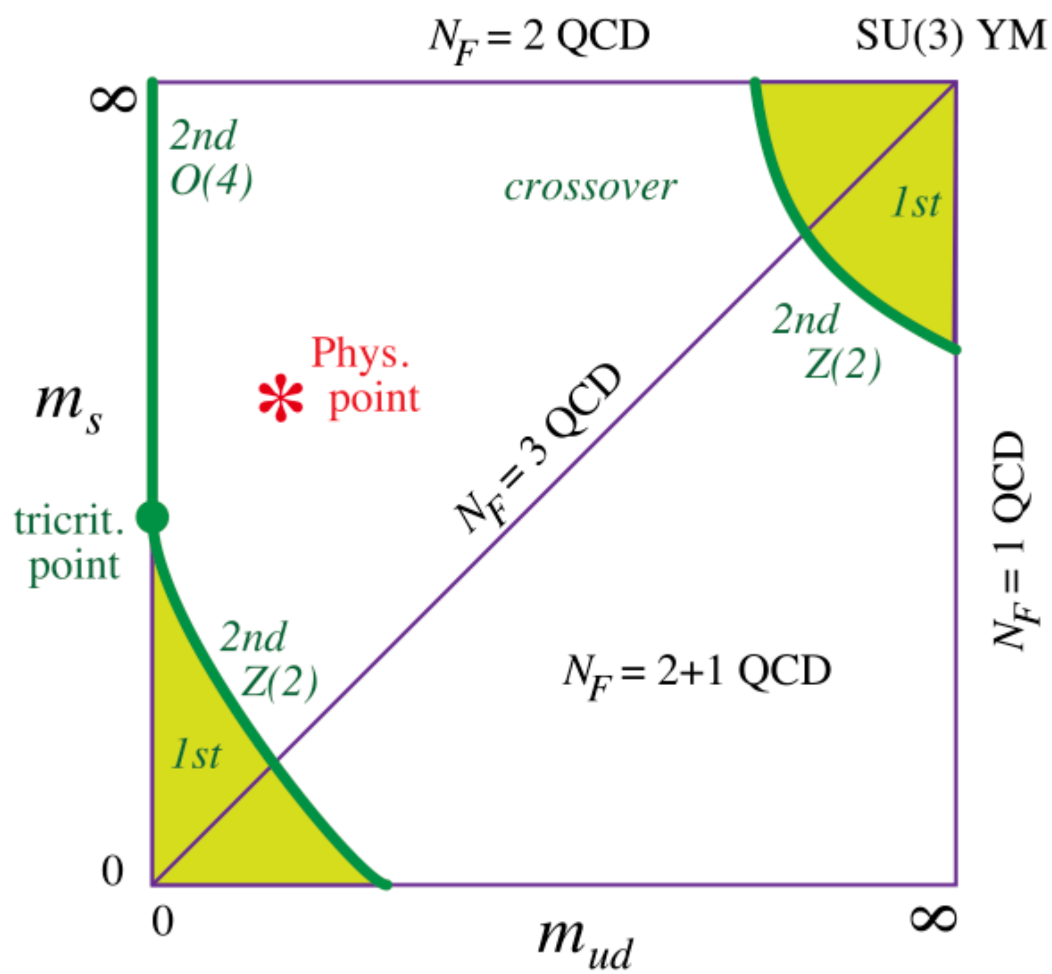
Big Bang

≈ **+0.1 m sec** (10^{12} K)
Phase transition:
hot quark matter (QGP)
=> ordinary matter (protons etc.)

Important as the initial condition of our Universe (atoms, molecules, stars, life, ...).

Nature of $T > 0$ QCD transition as function of m_q's

The traditional picture given by this Columbia plot

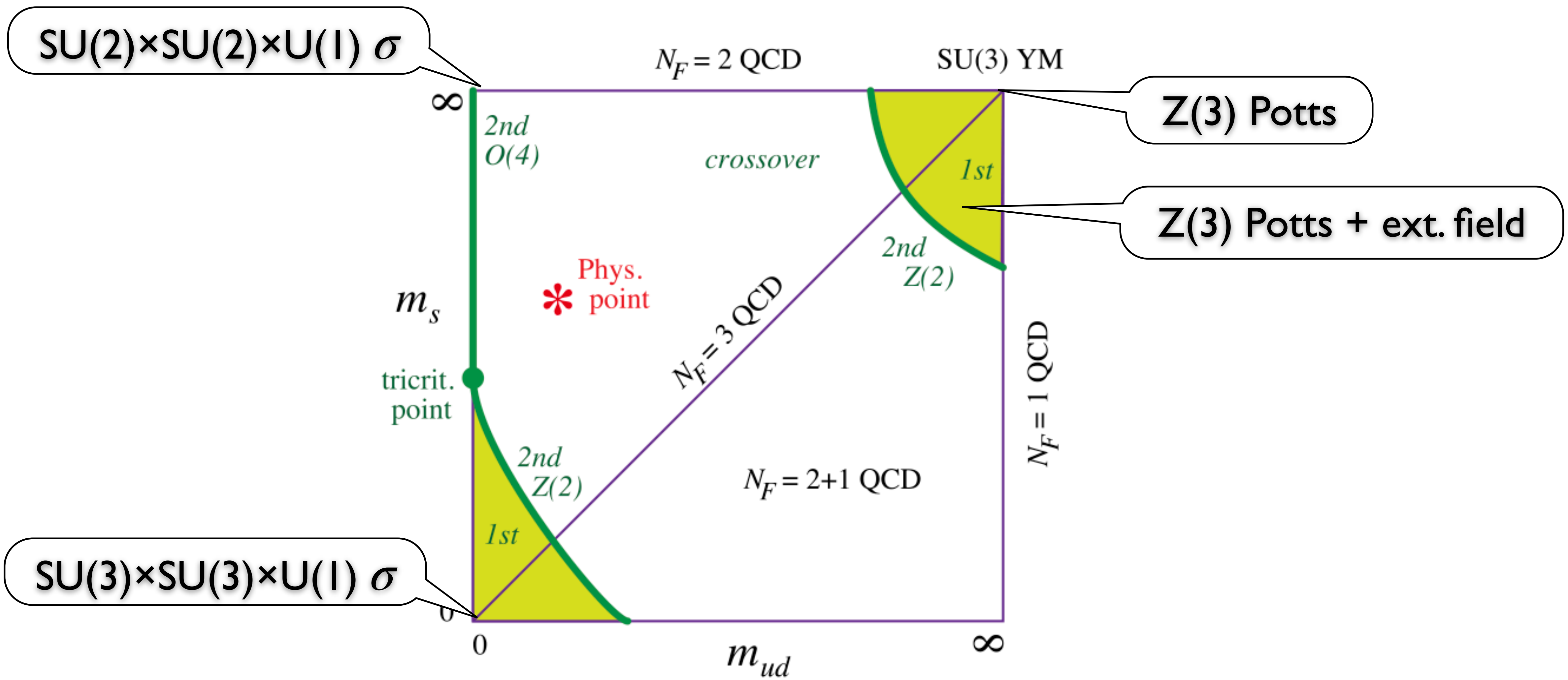


- The physical point locates in the crossover region.

Nature of the QCD transition off the phys. pt. is important
<= properties at the phys. pt. affected by scaling of nearby critical points.

Nature of $T > 0$ QCD transition as function of m_q's

The traditional picture given by this Columbia plot

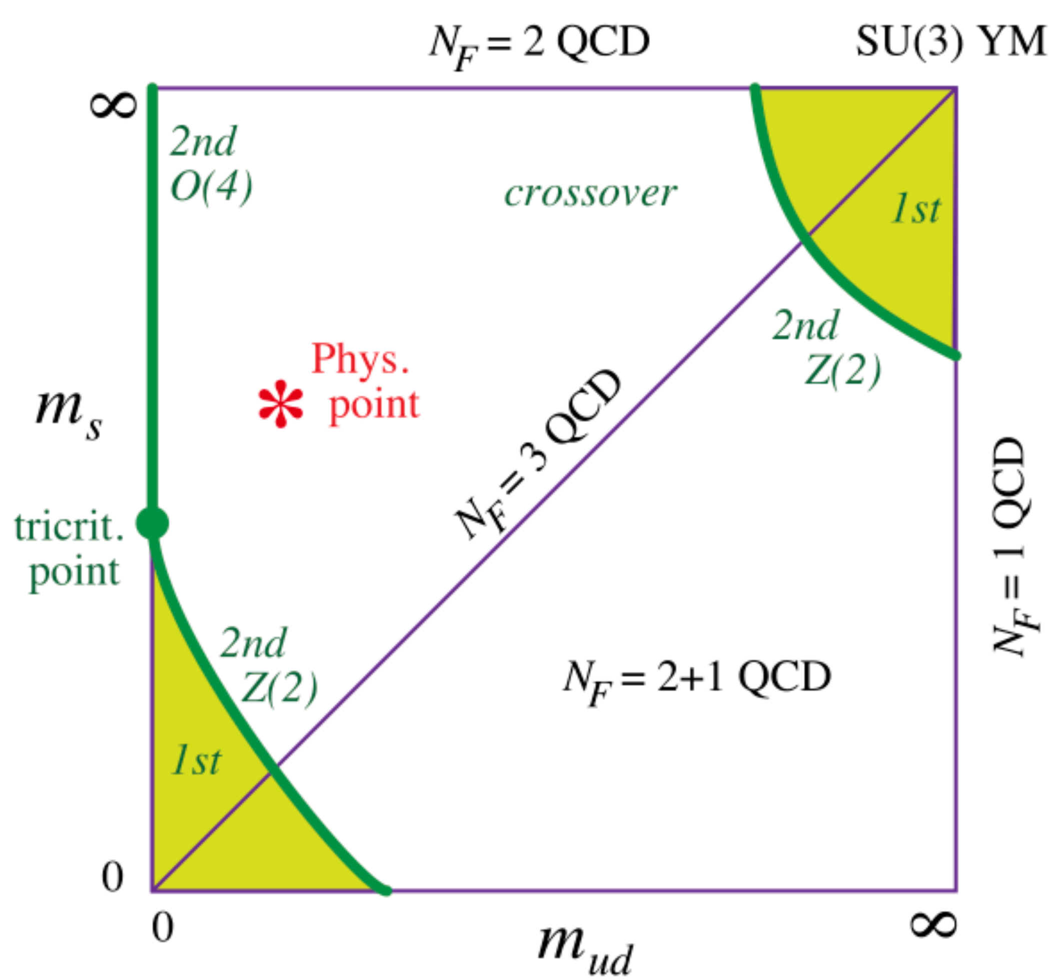


- **Universality arguments with effective chiral σ model around the chiral limits**
  - Pisarski-Wilczek, Phys. Rev. D 29 (1984) 338
  - Wilczek, Int. J. Mod. Phys. A7 (1992) 3911
  - Rajagopal-Wilczek Nucl. Phys. B399 (1993) 395

=> $N_f = 2$: 2nd order in the O(4) univ. class // $N_f \geq 3$: 1st order

Nature of $T > 0$ QCD transition as function of m_q's

The traditional picture given by this Columbia plot ***is still under many discussions.***



- If $U(1)_A$ [broken by anomaly at all *T*'s] is *effectively* restored around *Tc*

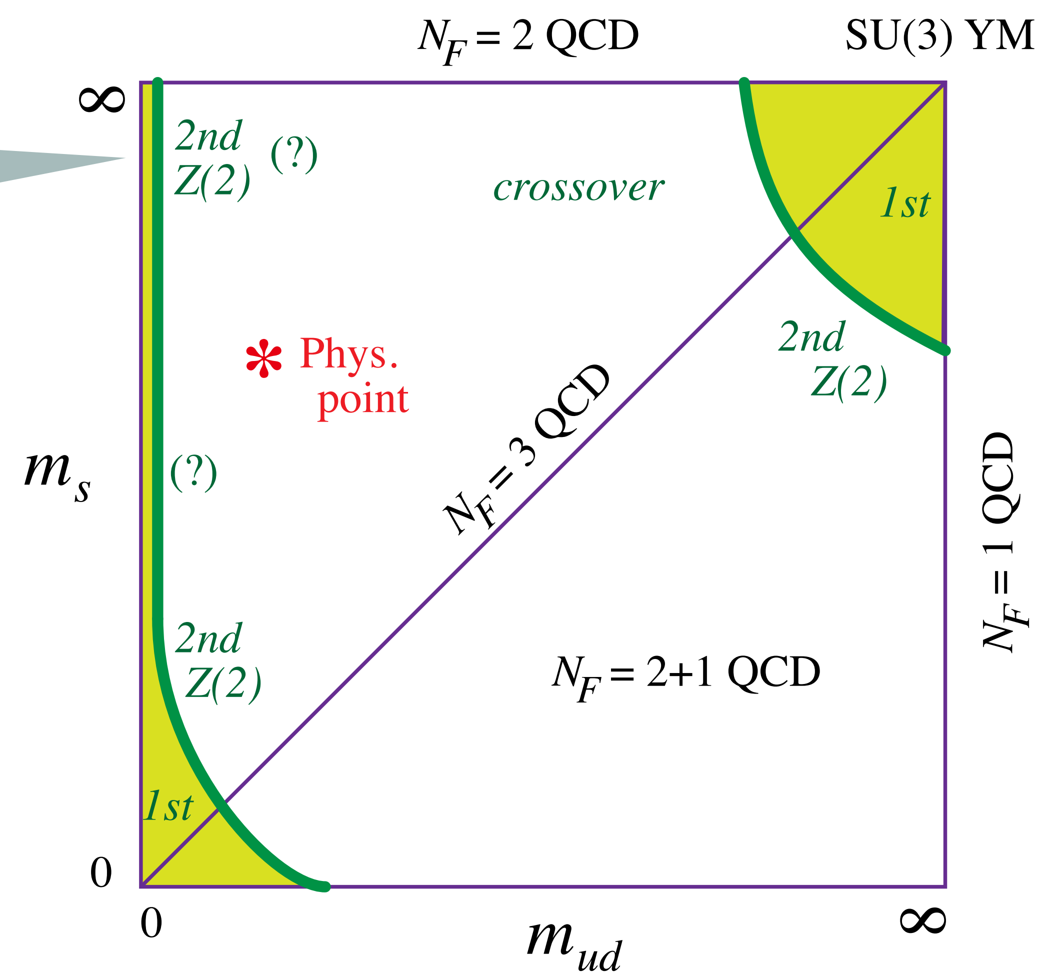
=> $N_f = 2$ chiral trans. is either 2nd or 1st

=> $N_f = 3$ 1st order region may be smaller [anomaly was a source of ϕ^3pot.]

Nature of $T > 0$ QCD transition as function of m_q's

The traditional picture given by this Columbia plot is still under many discussions…

Is the 2-flavor chiral limit 2nd order?



- 1st order for $N_f = 2$ on coarse lattices with unimproved lattice quarks
  - D'Elia+, PRD 72 (2005); Cossu+, Lattice2008; Philipsen+Pinke, PRD 93 (2016)

  See also
  - S.Aoki+, PRD 86 (2012)

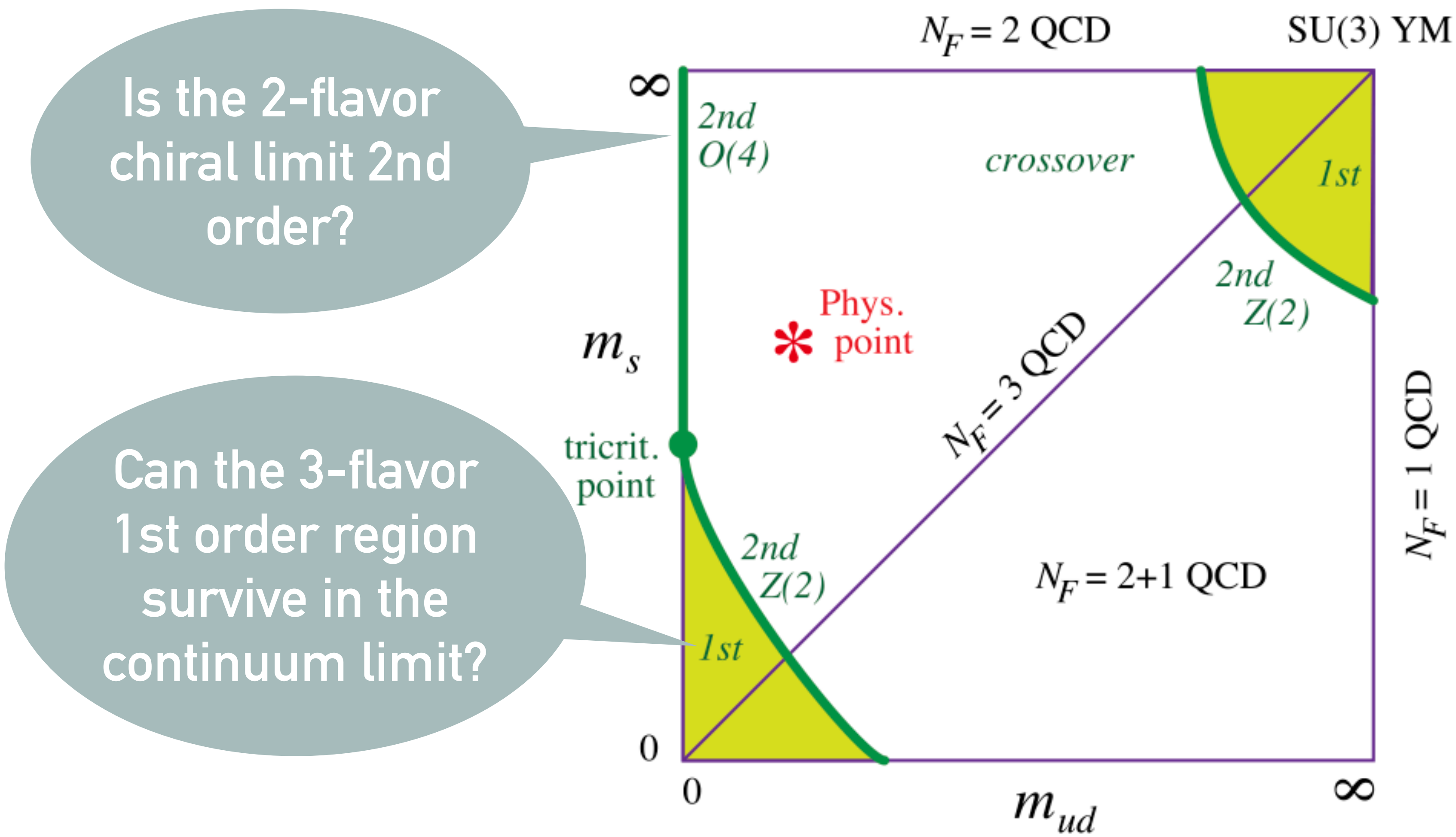
- If U(1)A [broken by anomaly at all T's] is *effectively* restored around Tc

  => $N_f = 2$ chiral trans. is either 2nd or 1st

  => $N_f = 3$ 1st order region may be smaller [anomaly was a source of ϕ^3pot.]

Nature of $T > 0$ QCD transition as function of m_q's

The traditional picture given by this Columbia plot is still under many discussions…



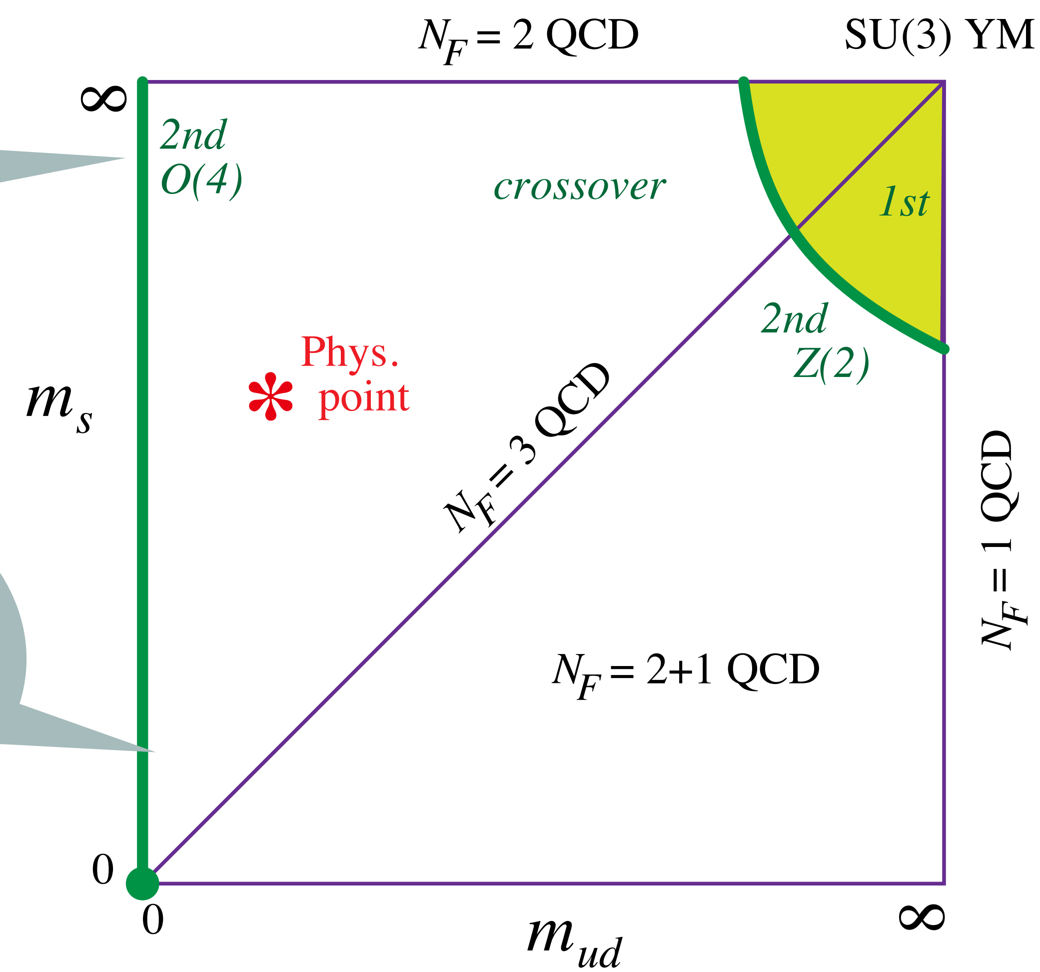
- Light-quark 1st order region for $N_f \approx 3$ shrinks with $a \to 0$ (Nt $\to \infty$)
  - Kuramashi+, *PRD* 101 (2020): improved Wilson, Nt=12
  - Dini+, *PRD* 105 (2022): improved KS, Nt=8

Nature of $T > 0$ QCD transition as function of m_q's

The traditional picture given by this Columbia plot is still under many discussions…

Is the 2-flavor chiral limit 2nd order?

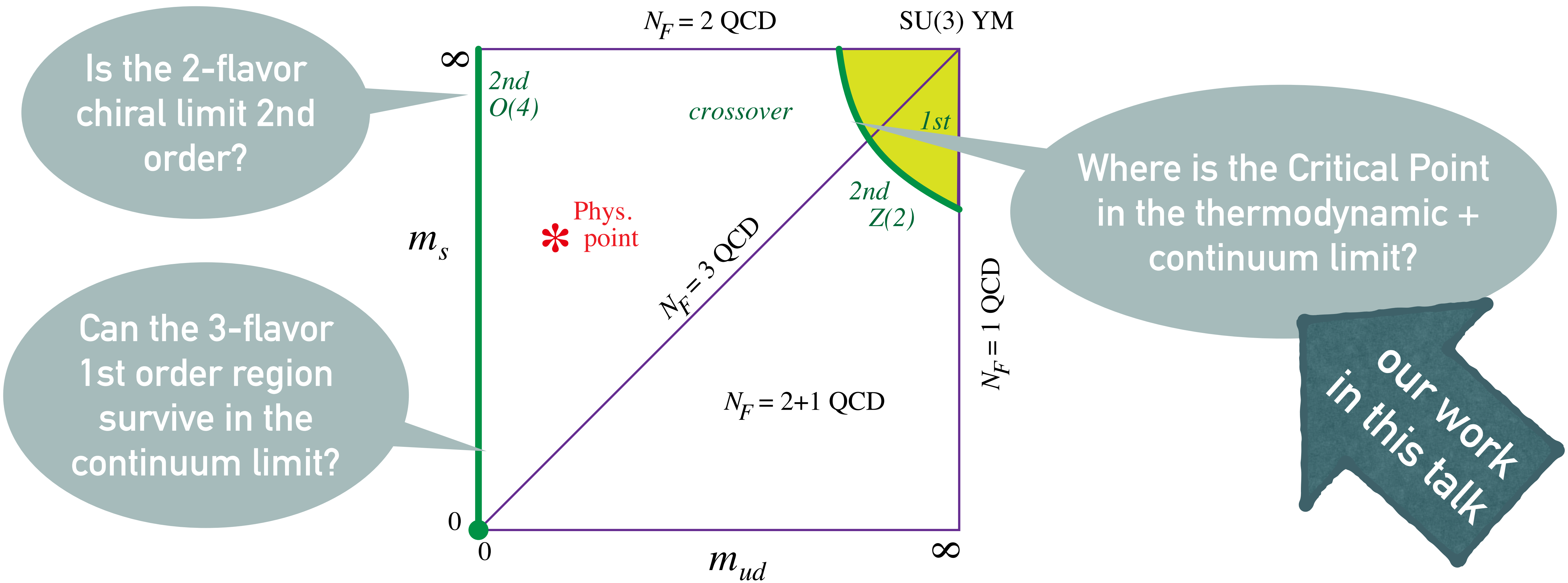
Can the 3-flavor 1st order region survive in the continuum limit?



- Light-quark 1st order region for $N_f \approx 3$ shrinks with $a \to 0$ (Nt $\to \infty$)
  - Kuramashi+, *PRD* 101 (2020): improved Wilson, Nt=12
  - Dini+, *PRD* 105 (2022): improved KS, Nt=8

Nature of $T > 0$ QCD transition as function of m_q's

The traditional picture given by this Columbia plot is still under many discussions…



- Recent studies on the location of CP in heavy-quark QCD
  - Saito+ (WQHOT-QCD), *PRD* (2011/2014): HPE LO, Nt=4, Ns/Nt=6
  - Ejiri+ (WHOT-QCD), *PRD* (2020): HPE eff-NLO, Nt=6, Ns/Nt=4–6; Nt=8, Ns/Nt=3
  - Cuteri+, *PRD* (2021): Nf=2, fullQCD, Nt=6,8,10, Ns/Nt=4–7(10)

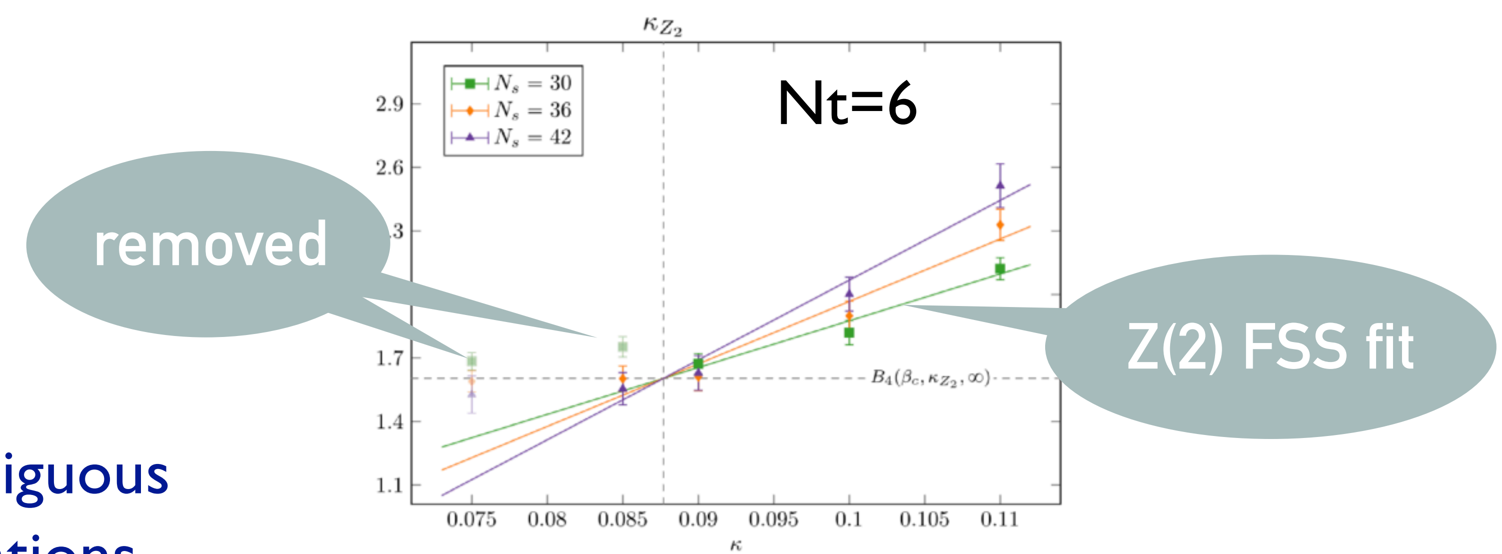
=> We still have strong cutoff & spatial volume dependences.

Motivations

- Binder cumulant analysis based on the Z(2) FSS expected around CP

So far, however, identification of the Z(2) FSS is not a simple task
--- removal of many high-T data required / correction terms to the FSS introduced.

These make the analyses slightly ambiguous & call careful systematic error estimations.



(a) Fit e.6.1 at $N_\tau = 6$.

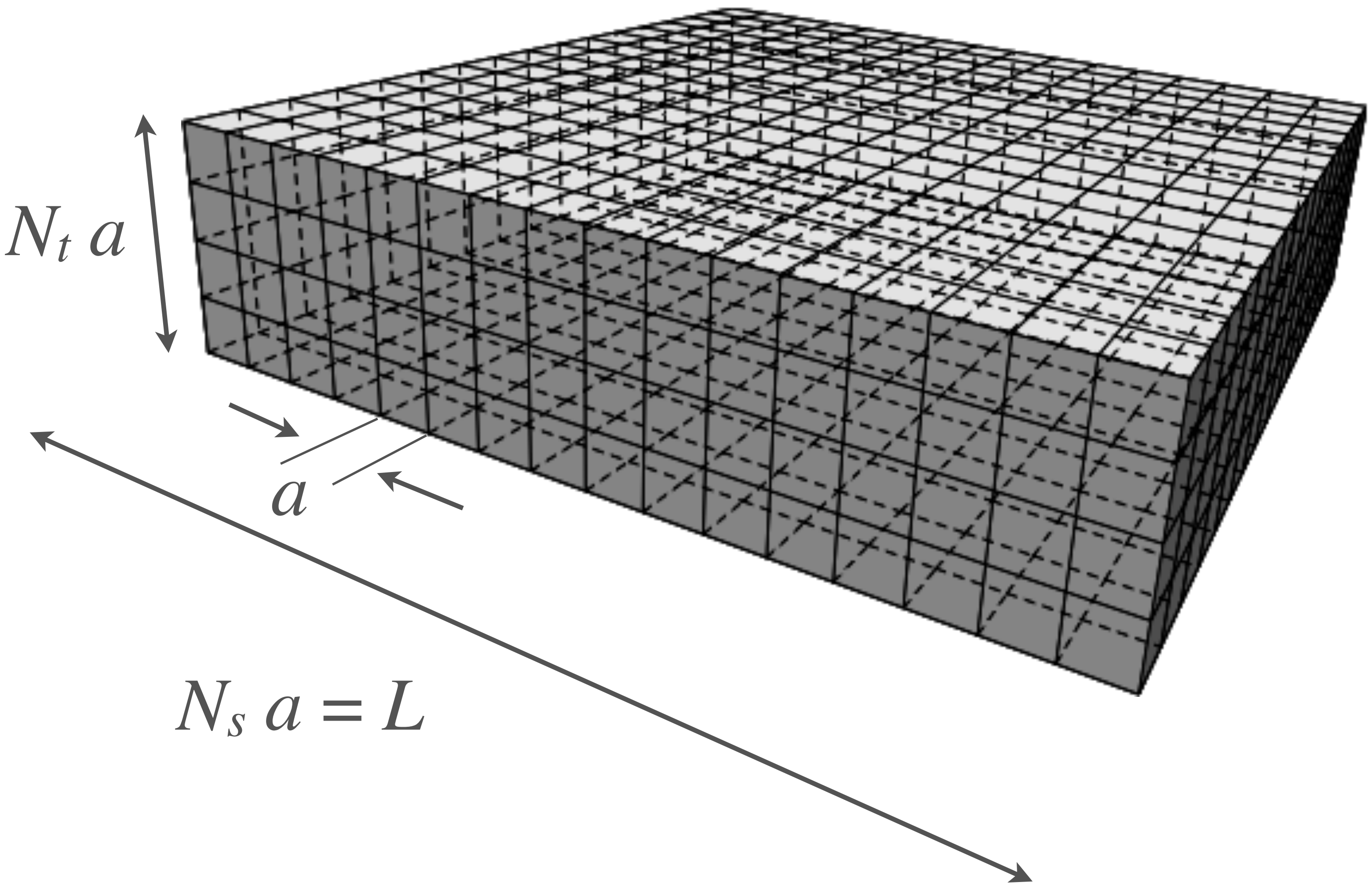
Cuteri+, *PRD* (2021)

=> Simulations with larger spatial volumes & high statistics to identify the FSS more clearly.
=> Multi-point reweighting to vary coupling parameters continuously.

This talk is based on

- Kiyohara+ (WQHOT-QCD), *Phys.Rrev.D* (2021) [DOI: 10.1103/PhysRevD.104.114509]
- Wakabayashi+ (WHOT-QCD), *Prog.Theor.Exp.Phys.* (2022) [DOI: 10.1093/ptep/ptac019]
- Ashikawa+ (WHOT-QCD), *ongoing*

Lattice setup



$$T = \frac{1}{N_t a}$$

N_t : lattice size in the euclidian time direction

a : lattice spacing

For a given T,
continuum limit: $N_t \to \infty$

Spatial system size L = Ns a <= we control it by the "aspect ratio" Ns / Nt = LT.

- previous studies : $LT \approx 4 - 7$ (10)
- our sudy : LT up to 12 or 15

To suppress compt. costs on large spatial volumes, we first revisited Nt=4 [Kiyohara+].
We are now extending the study to Nt=6 [Ashikawa+, ongoing].

Lattice setup

- Our lattice action: plaquette gauge + standard Wilson quarks
- Wilson quark kernel:

$$M_{xy}(\kappa) = \delta_{xy} - \kappa \sum_{\mu} \left[(1-\gamma_\mu) U_{x,\mu} \delta_{y,x+\hat{\mu}} + (1+\gamma_\mu) U^\dagger_{y,\mu} \delta_{y,x-\hat{\mu}} \right]$$

$$= \delta_{xy} - \kappa B_{xy}$$

hopping term

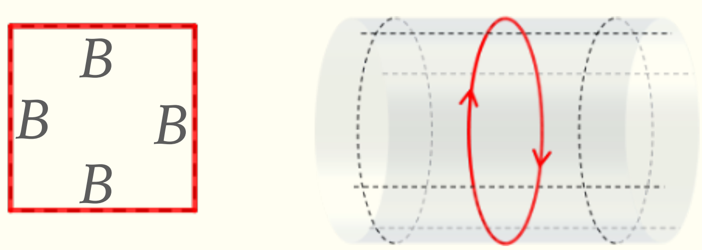
$$\kappa = \frac{1}{2am_q + 8}$$

- Quark contribution to the effective action: $\ln \det M(\kappa) = -\frac{1}{N_{\mathrm{site}} n} \sum_{n=1}^{\infty} \mathrm{Tr}[B^n] \kappa^n$
  - closed loops of B with $\kappa^{\text{[loop length]}}$

- Hopping Parameter Expansion to reduce simulation costs for large spatial volumes

LO:

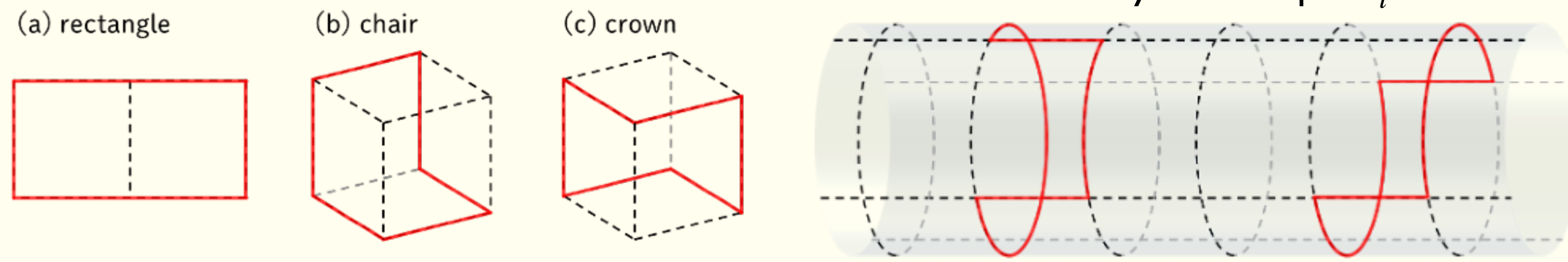
plaquette | Polyakov loop Ω



κ^4 | κ^{N_t}

NLO:

(a) rectangle | (b) chair | (c) crown | bent Polyakov loops Ω_i



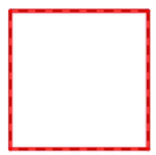
κ^6 | κ^{N_t+2}

- HPE $\approx 1/(am_q)$ expansion
- HPE worsens with $a \to 0$ $(N_t \to \infty)$ => higher order terms required with $N_t \to \infty$.

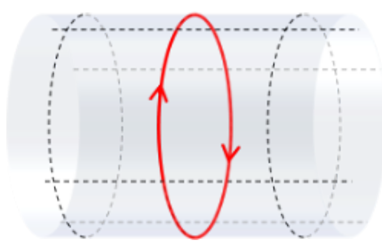
Simulation incorporating LO + NLO meas.'s

Kiyohara, Kitazawa, Ejiri, KK (WHOT-QCD), *PRD* 104 (2021)

- LO incorporated in the configuration generation



$$\beta \;\rightarrow\; \beta^* = \beta + 48 N_f \kappa^4$$



$\lambda \sum_{\mathbf{x}} \Omega(\mathbf{x})$ term in the effective action ($\lambda = 48 N_f N_t \kappa^4$ for Nt=4)

can be incorporated in PHB+OR parallel simulation efficiently

by keeping all temporal sites within a node

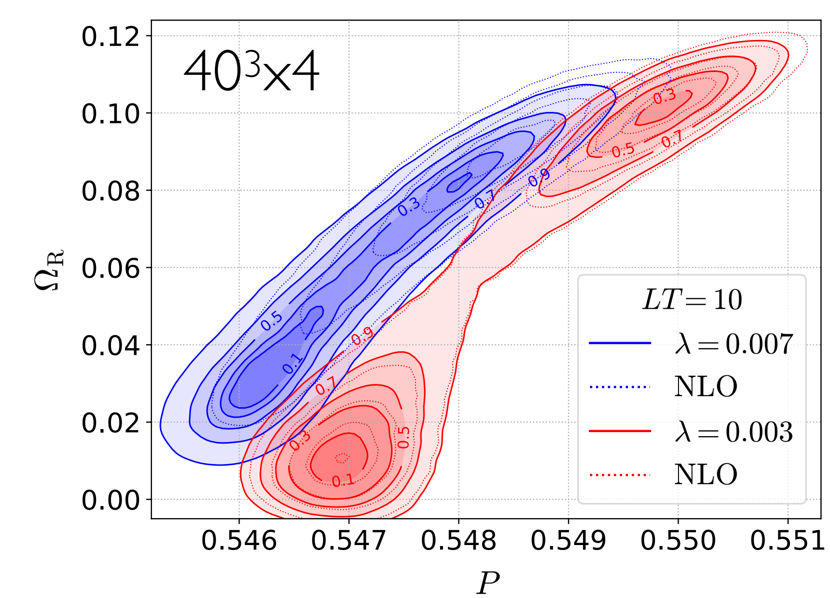
=> Simulation cost ≤ 1/100 of full-QCD simulations : large spatial volume enabled

- NLO incorporated in the measurements through multi-point reweighting

$$\langle \hat{O}(U) \rangle^{\rm NLO}_{\beta,\lambda} = \frac{\langle \hat{O}(U)\, e^{-\delta S_{\rm LO} - S_{\rm NLO}(\beta,\lambda)} \rangle^{\rm LO}_{\tilde{\beta},\tilde{\lambda}}}{\langle e^{-\delta S_{g+\rm LO} - S_{\rm NLO}(\beta,\lambda)} \rangle^{\rm LO}_{\tilde{\beta},\tilde{\lambda}}}$$

$$\delta S_{g+\rm LO} = S_{g+\rm LO}(\beta,\lambda) - S_{g+\rm LO}(\tilde{\beta},\tilde{\lambda})$$

- Simulations at several $(\tilde{\beta}^*, \tilde{\lambda})$ => measure at (β^*, λ)
- Overlap problem resolved by the inclusion of LO in configuration generations <= essential on spatially large lattices in this study

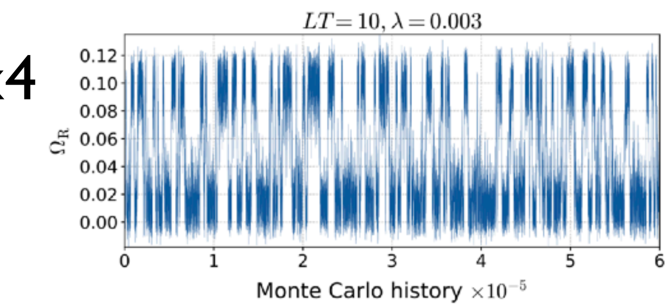


Study on $N_t = 4$ lattices

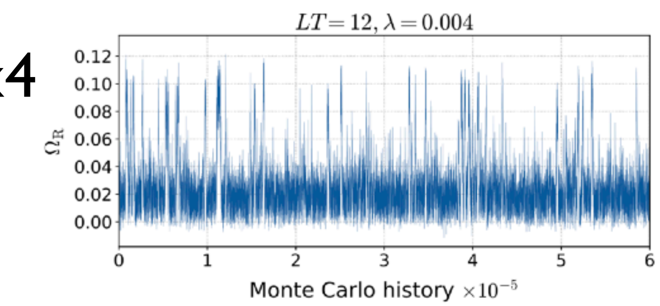
Kiyohara, Kitazawa, Ejiri, KK (WHOT-QCD), *PRD* 104 (2021)

- Simulations: Nt=4, Ns/Nt = LT = 6, 8, 9, 10, 12, each 3-6 x [$(\tilde{\beta}^*, \tilde{\lambda})$ with ~10^6 meas.] around the transition line L = spatial lattice size, $\lambda = 48 N_f N_t \kappa^4$ for Nt=4

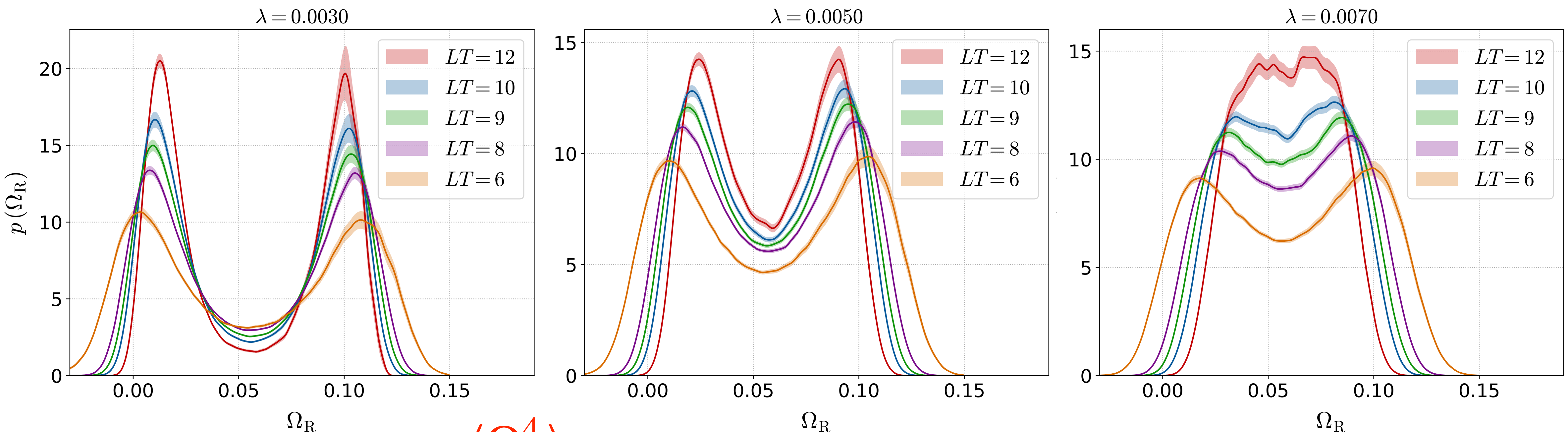
- History of $\Omega_R = \mathrm{Re}\Omega$ 40^3x4



48^3x4



- Distribution of Ω_R on the transition line

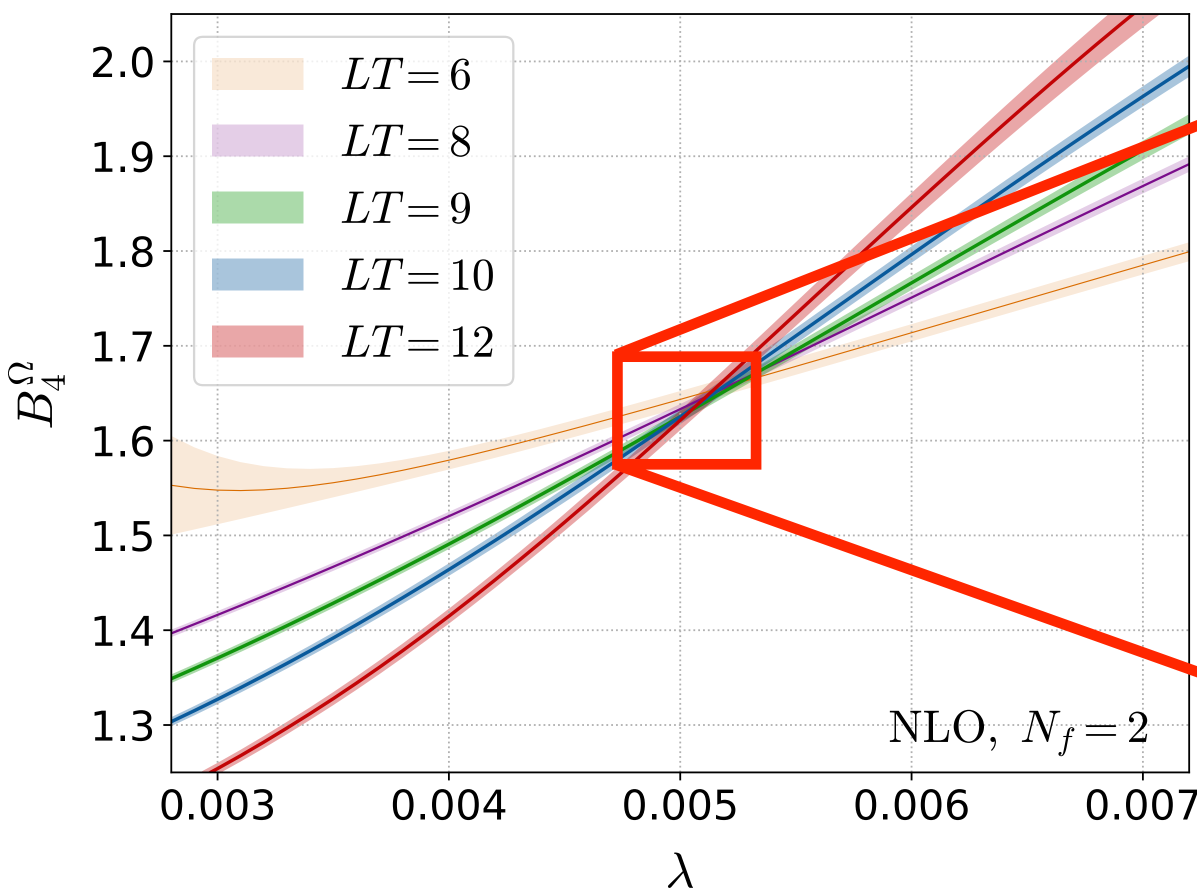


=> Binder cumulant $B_4^\Omega = \frac{\langle \Omega_R^4 \rangle_c}{\langle \Omega_R^2 \rangle_c^2} + 3$ along the transition line in the (β, κ) plane.

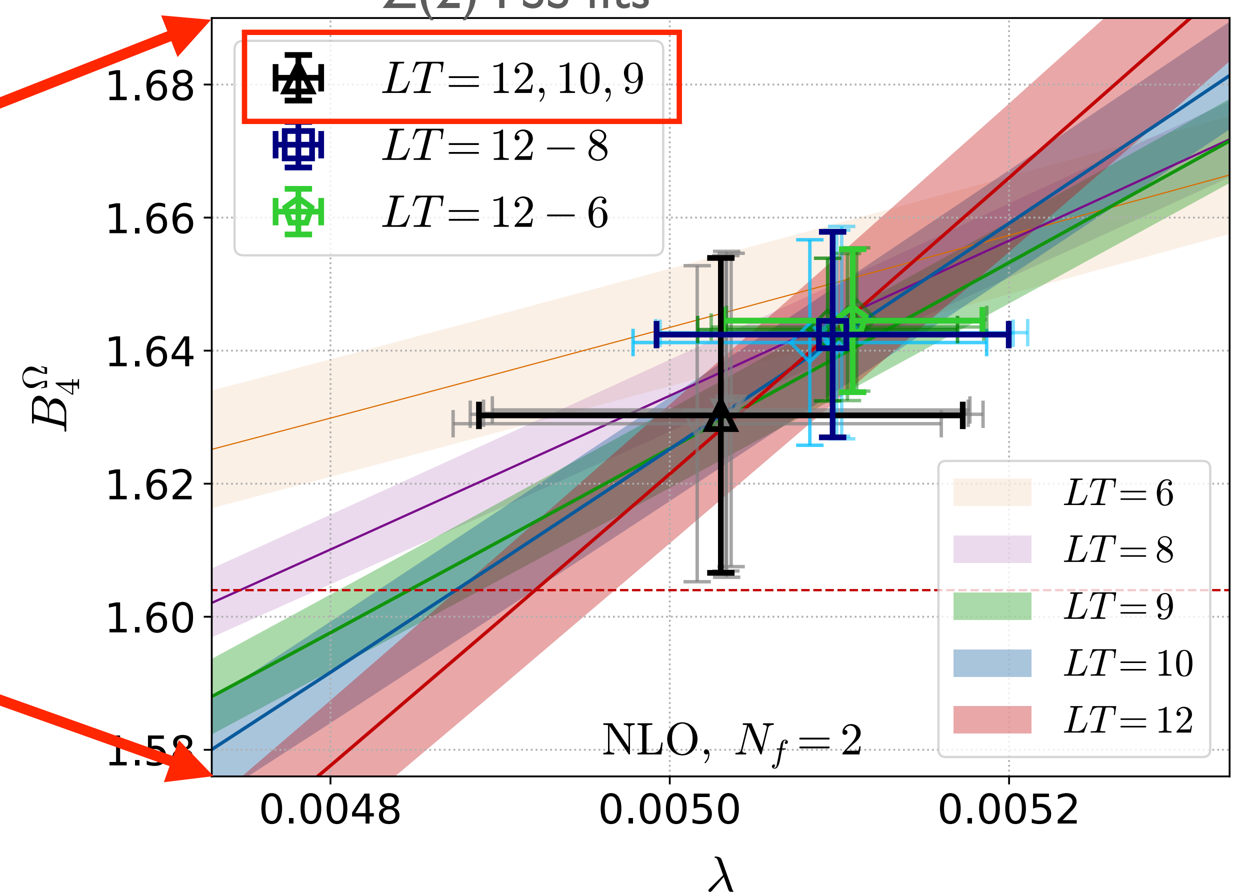
Study on $N_t = 4$ lattices

Kiyohara, Kitazawa, Ejiri, KK (WHOT-QCD), *PRD* 104 (2021)

- Results at Nt=4 with HPE up to NLO



Z(2) FSS fits



- ☆ Precision much improved over previous studies
- ☆ Ns/Nt = LT ≥ 9 required for Z(2) FSS
- ☆ $B_4^\Omega = 1.630(24)(2)$ using Ns/Nt ≥ 9, consistent with Z(2) value 1.604 within $\approx 1\sigma$
- ☆ $\lambda_c = 0.00503(14)(2)$ [$\kappa_c = 0.0603(4)$] for Nt=4, Nf=2

(cf.) Ejiri+ PRD(2020): $\kappa_c = 0.0640(10)$ with eff. NLO

Scope and convergence of HPE

Wakabayashi, Ejiri, KK, Kitazawa (WHOT-QCD), *PTEP* (2022)

Are the effects of further higher orders of HPE really negligible?

- Quark contribution to the effective action:

loops of length n

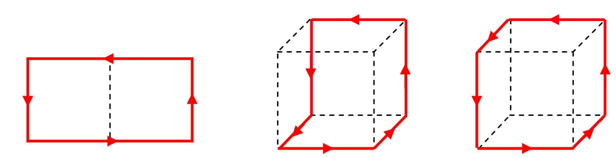
noise average

$$\ln \det M(\kappa) = N_{\rm site} \sum_n D_n \kappa^n, \qquad D_n = \frac{-1}{N_{\rm site} n} {\rm Tr}[B^n] \approx \frac{-1}{N_{\rm site} n} \langle\langle \eta^\dagger B^n \eta \rangle\rangle_{\rm noises}$$

$$B_{xy} = \sum_\mu \left[(1-\gamma_\mu) U_{x,\mu} \delta_{y,x+\hat\mu} + (1+\gamma_\mu) U^\dagger_{y,\mu} \delta_{y,x-\hat\mu} \right]$$

Wilson-type loops

$$W(4) = 96 N_c \hat{P}, \quad W(6) = 256 N_c (3\hat{W}_{\rm rec} + 6\hat{W}_{\rm chair} + 2\hat{W}_{\rm crown})$$

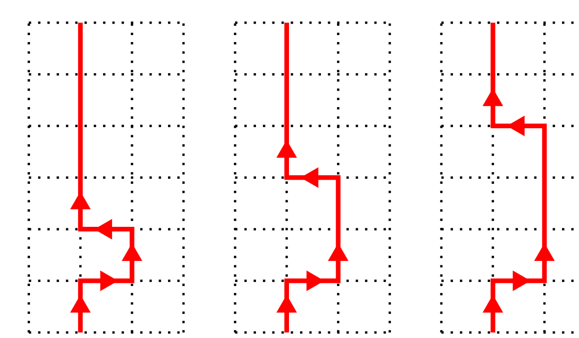


$$D_n = W(n) + \sum_m L_m(N_t, n) = W(n) + L(N_t, n)$$

Polyakov-type loops with m-windings

$$L_1(N_t, N_t) = \frac{4 N_c 2^{N_t}}{N_t} {\rm Re}\hat{\Omega}$$

$$L_1(N_t, N_t+2) = 12 N_c 2^{N_t} \sum_k {\rm Re}\hat{\Omega}_k$$



We developed a method to separately evaluate $W(n)$ and $L_m(N_t, n)$ from D_n by combing the results with various twisted boundary conditions.

Scope and convergence of HPE

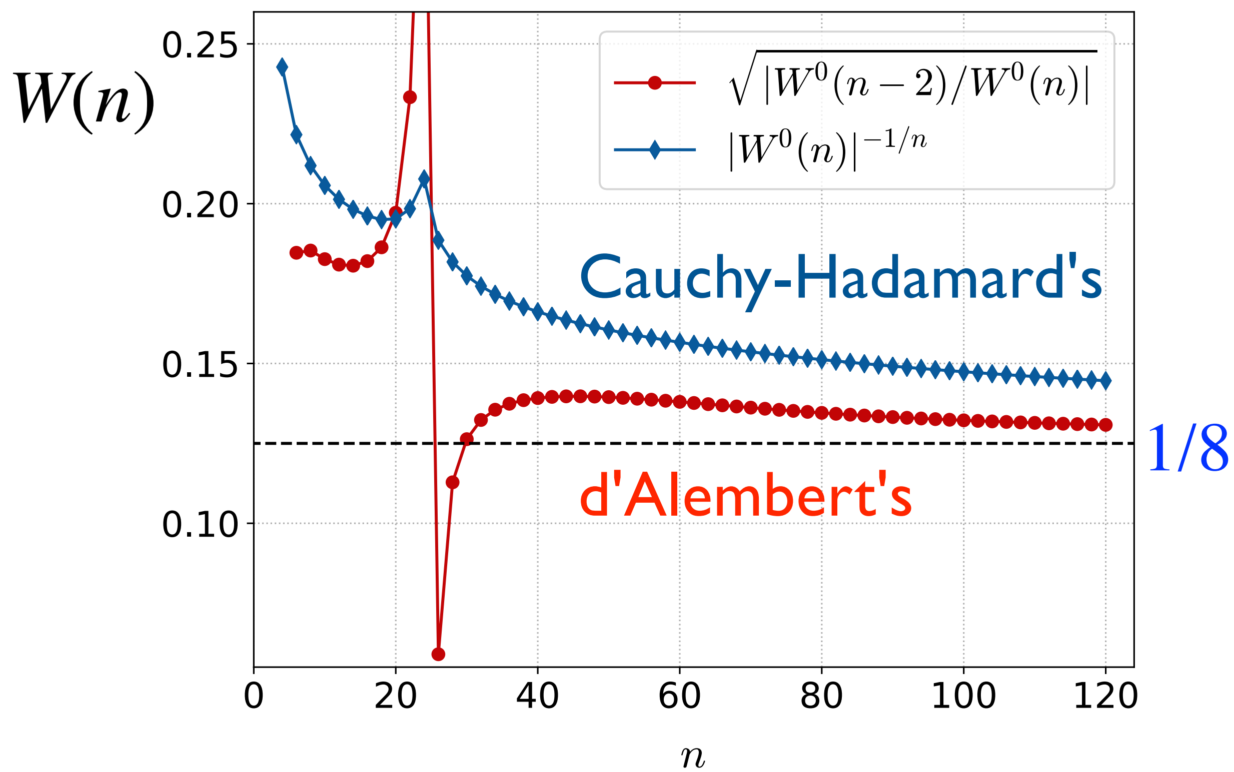
Wakabayashi, Ejiri, KK, Kitazawa (WHOT-QCD), *PTEP* (2022)

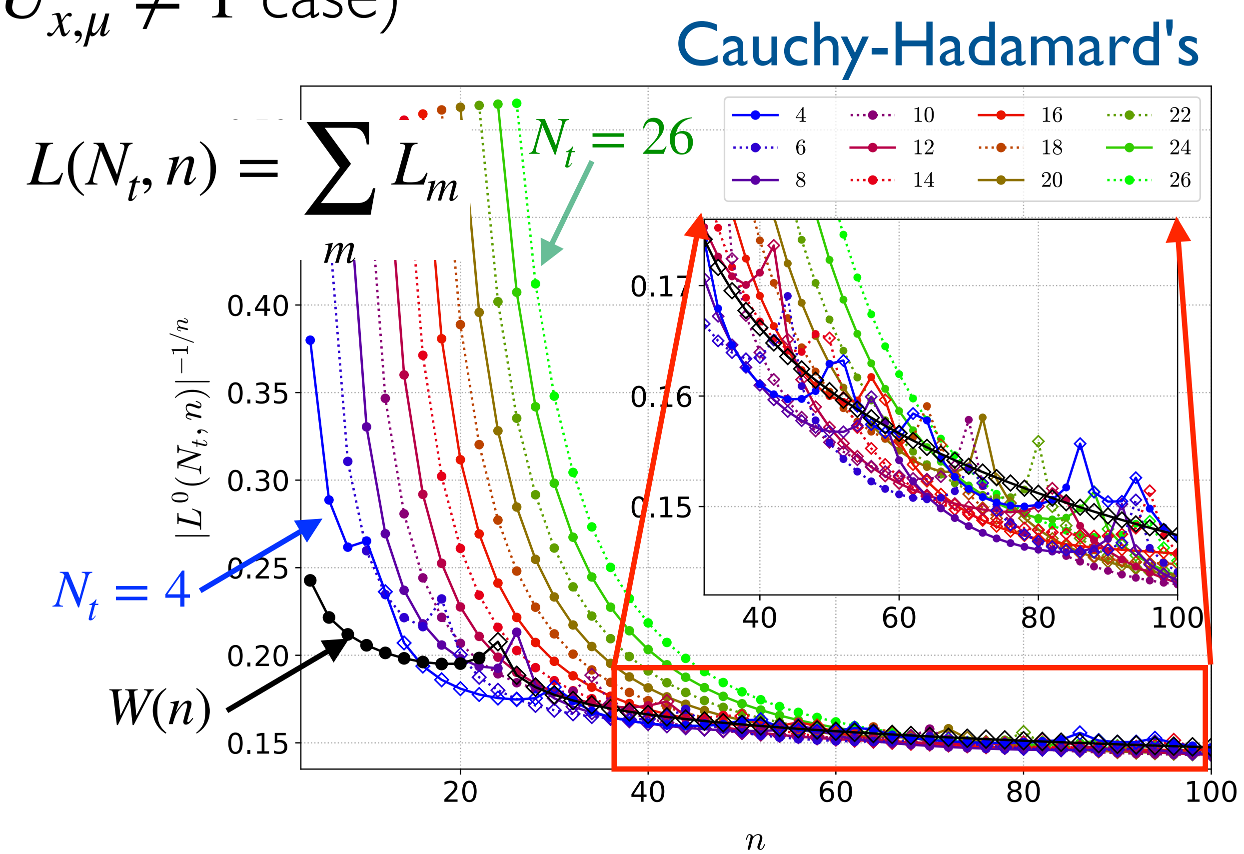
- $\hat{W}_i, \hat{P}_j$ in $W(n)$ and $L_m(N_t, n)$ take their maximum value 1 when we set $U_{x,\mu} = 1$

  In this case, we can calculate $W(n)$ and $L_m(N_t, n)$ analytically up to high orders.

=> Worst convergent case of HPE can be studied by combining them.

- **Convergence radius** (lower bound for the $U_{x,\mu} \neq 1$ case)





☆ Convergence radius $\xrightarrow[n \to \infty]{} 1/8$, i.e. convergent up to the chiral limit.

<= free Wilson quarks when $U_{x,\mu} = 1$

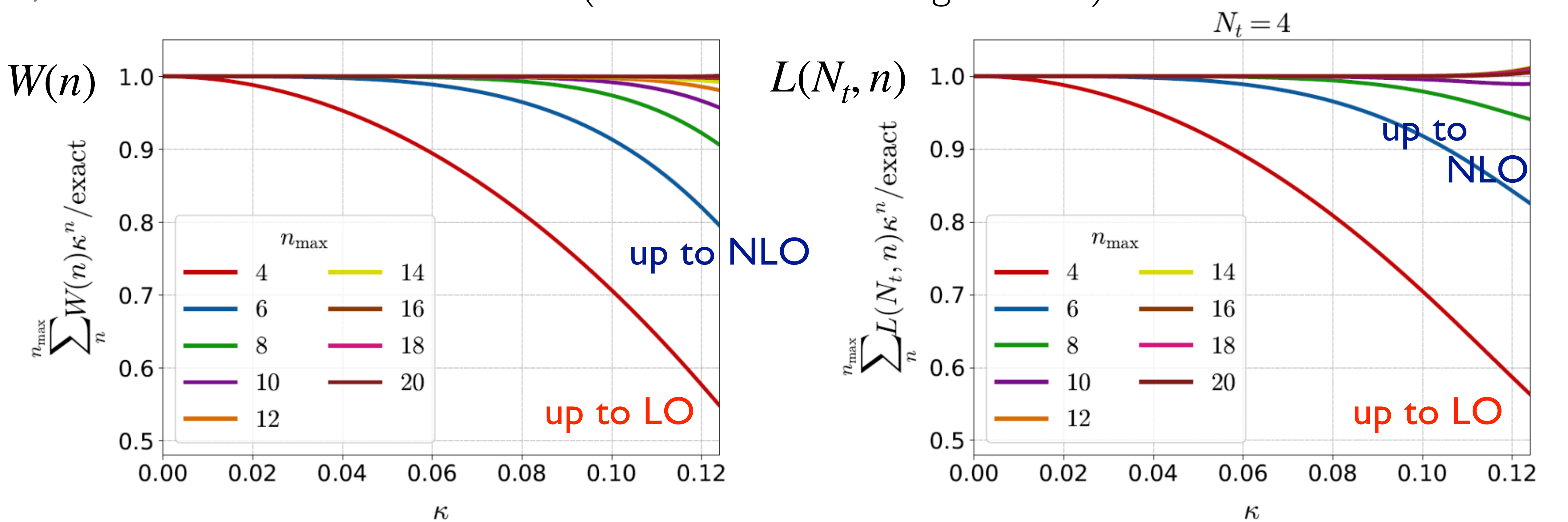
=> **HPE reliable at any m_q, when sufficiently high orders are taken.**

Scope and convergence of HPE

Wakabayashi, Ejiri, KK, Kitazawa (WHOT-QCD), *PTEP* (2022)

To which order we need to incorporate? <= *depends on the value of* κ

- **Deviation due to truncation** (in the worst convergent case):



- ☆ For Nt=4: $\kappa_c = 0.0603(4)$ [Kiyohara+ ('21)]
  => LO may have at worst ≈10% error, NLO good enough
- ☆ For Nt=6: $\kappa_c = 0.0877(9)$ [Cuteri+ ('22)], $0.1286(40)$ [Ejiri+ ('20) using eff. pot.]
  => NLO is ≥93% accurate. remaining error can be removed by NNLO or higher
- ☆ For Nt=8: $\kappa_c = 0.1135(8)$ [Cuteri+ ('22)] => NNLO needed for ≥95% accuracy

Effective method to incorporate high orders

Wakabayashi, Ejiri, KK, Kitazawa (WHOT-QCD), *PTEP* (2022)

Calcuration of high order term becomes quickly difficult with increasing n.

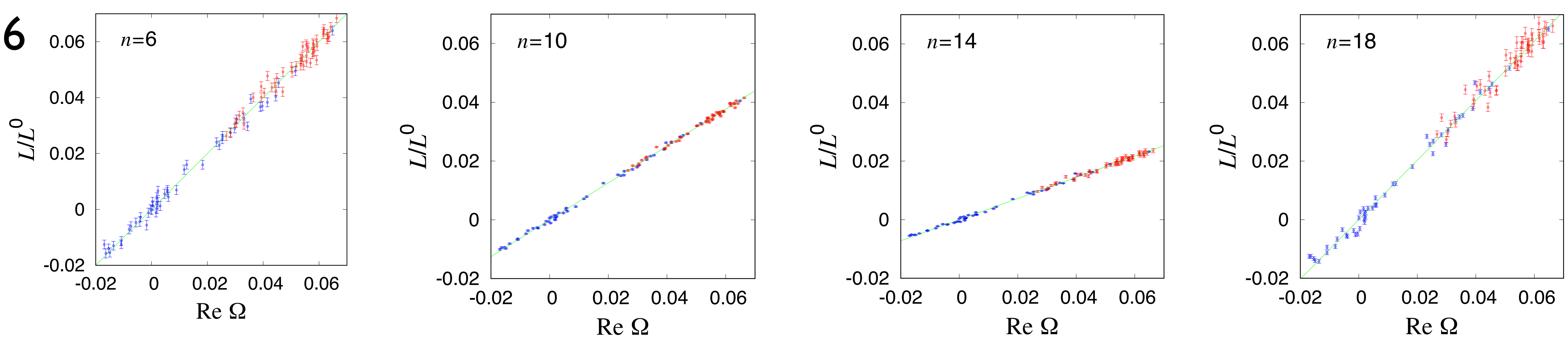
We extend the idea of the effective NLO method [Ejiri+ ('20)] to high orders.

Basic observation: **strong correlation of Wilson/Polyakov-type loops among different n.**

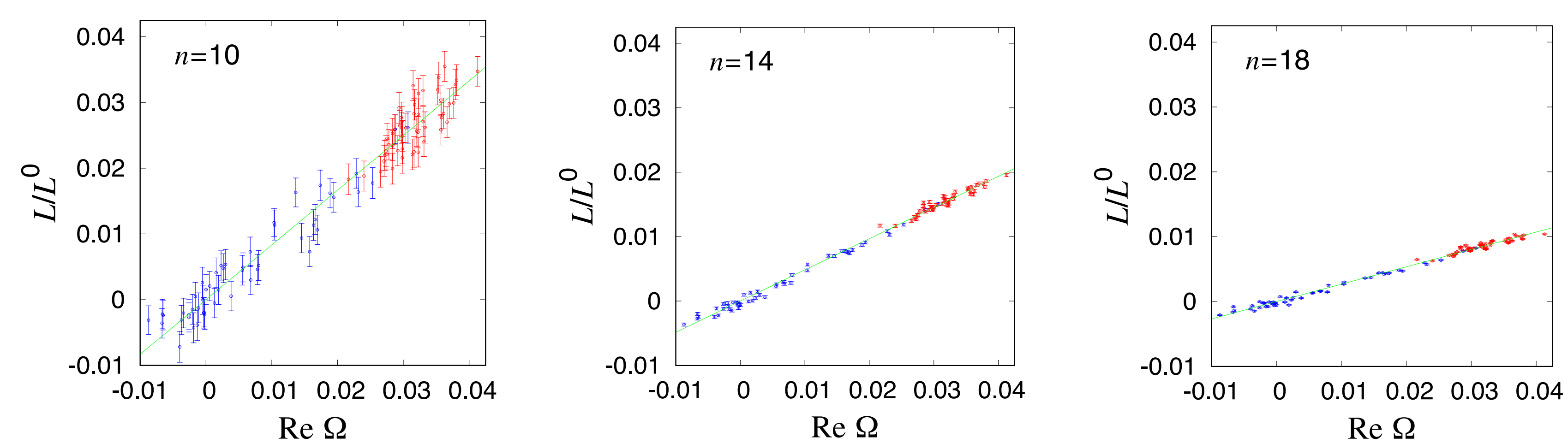
Distribution of $L(N_t, n)$ vs. the Polyakov loop Ω

- qQCD simulation on 32^3x(6, 8), blue/red slightly below/above β_{trans}
- normalized by the Uxμ=1 result L^0

Nt=6



Nt=8



Effective method to incorporate high orders

Wakabayashi, Ejiri, KK, Kitazawa (WHOT-QCD), *PTEP* (2022)

This linear correlation suggests us to approximate

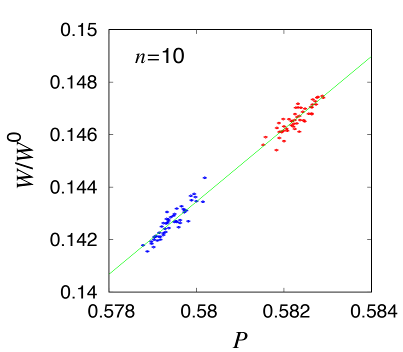
☆ $L(N_t, n) \approx L^0(N_t, n)\, c_n\, \mathrm{Re}\hat{\Omega}$

known from $U_{x\mu}=1$

slope measured by simulation

	$N_t = 6$	$N_t = 8$
c_6	1	
c_8	0.8112(20)(7)	1
c_{10}	0.6280(15)(3)	0.8327(114)(95)
c_{12}	0.4736(29)(15)	0.6408(36)(27)
c_{14}	0.3609(26)(11)	0.4841(22)(10)
c_{16}	0.3106(25)(10)	0.3616(21)(6)
c_{18}	1.0159(90)(33)	0.2679(16)(3)
c_{20}	−0.02771(57)(13)	0.2020(13)(2)

☆ $W(n) \approx W^0(n)\,(d_n\hat{P} + f_n)$



though the correlation weaker than L(Nt,n)

n	$d_n(N_t = 6)$	$f_n(N_t = 6)$	$d_n(N_t = 8)$	$f_n(N_t = 8)$
4	1	0	1	0
6	1.3625(73)(12)	−0.4070(42)(7)	1.3366(66)(8)	−0.3922(39)(5)
8	1.4644(123)(11)	−0.6089(72)(6)	1.4256(96)(8)	−0.5869(57)(5)
10	1.3835(156)(10)	−0.6590(91)(6)	1.3433(117)(8)	−0.6367(70)(5)
12	1.2140(178)(9)	−0.6235(103)(5)	1.1752(130)(7)	−0.6025(78)(4)
14	1.0256(196)(9)	−0.5533(114)(5)	0.9825(141)(7)	−0.5303(85)(4)
16	0.8607(219)(9)	−0.4811(127)(5)	0.8052(153)(8)	−0.4512(92)(5)
18	0.7481(258)(10)	−0.4296(150)(6)	0.6698(173)(9)	−0.3870(103)(5)
20	0.7290(337)(12)	−0.4275(196)(7)	0.6071(219)(12)	−0.3606(131)(7)

=> Higher order effects can be effectively incorporated in the LO simulation by

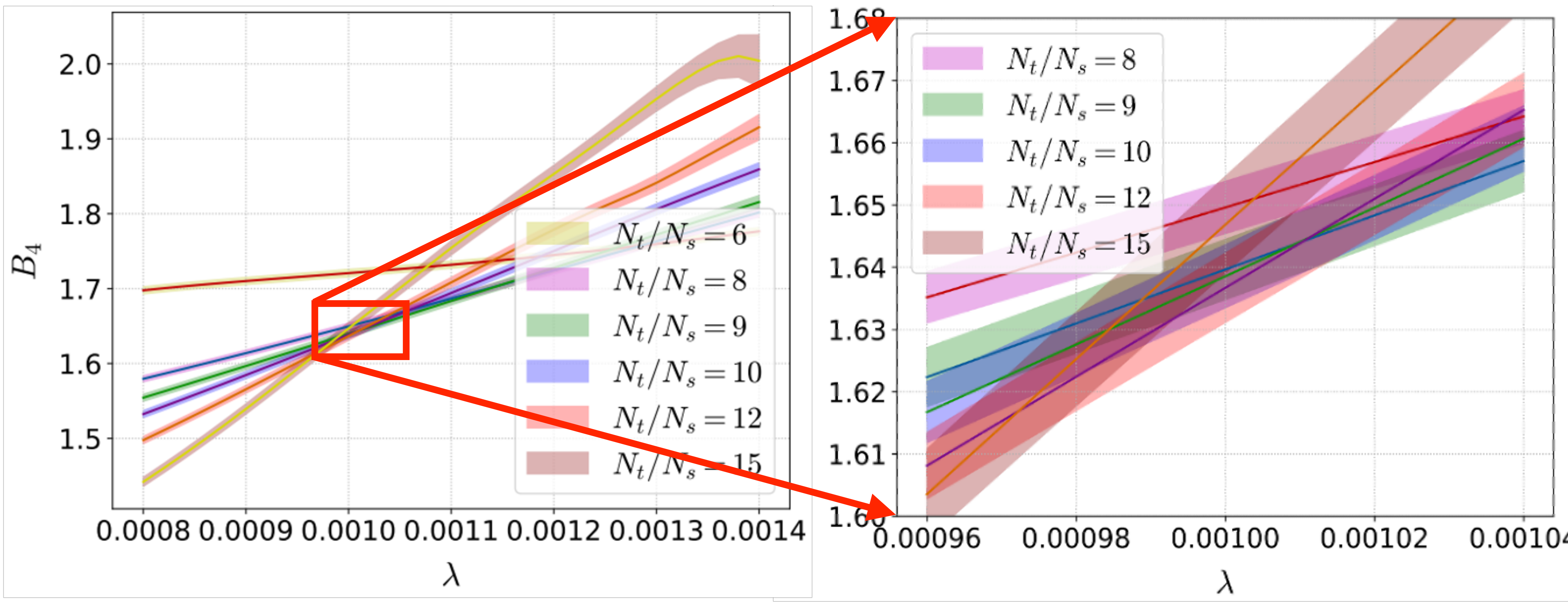
$$\beta \;\to\; \beta^* = \beta + \frac{1}{6} N_f \sum_{n=4}^{n_{\max}} W^0(n)\, d_n\, \kappa^n \qquad \lambda \;\to\; \lambda^* = N_f N_t \sum_{n=N_t}^{n_{\max}} L^0(N_t, n)\, c_n\, \kappa^n$$

Extension to non-degenerate cases (Nf=2+1 etc.) straightforward.

Study on $N_t = 6$ lattices

Ashikawa+ (WHOT-QCD), *ongoing*

- Nt=6, Ns/Nt = LT = 6, (7,) 8, 9, 10, 12, 15 *ongoing*
- Status of B_4^{Ω} with NLO: $\lambda = 128 N_f N_t \kappa^6$ for Nt=6, Nf=2, NLO



Preliminary:

☆ $B_4^{\Omega} \sim 1.61 - 1.62$ with Ns/Nt = 12–15 (cf.) Z(2) value = 1.604

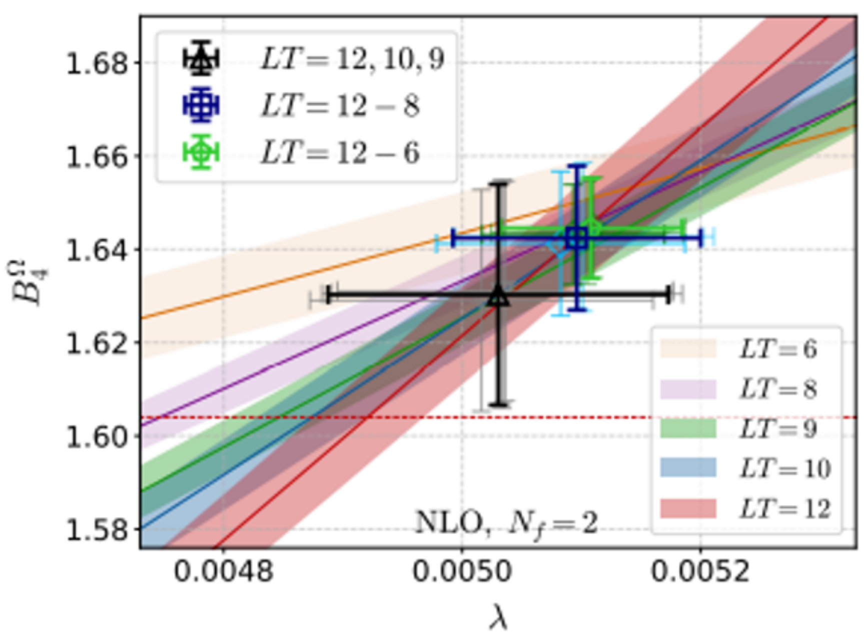
☆ $\lambda_c \sim 0.00097 - 0.00101$ => $\kappa_c \sim 0.093$ NLO => $\kappa_c \sim 0.0905$ up to 20th order

(cf.) $\kappa_c = 0.0877(9)$ by a full QCD simulation [Cuteri+ ('22)]

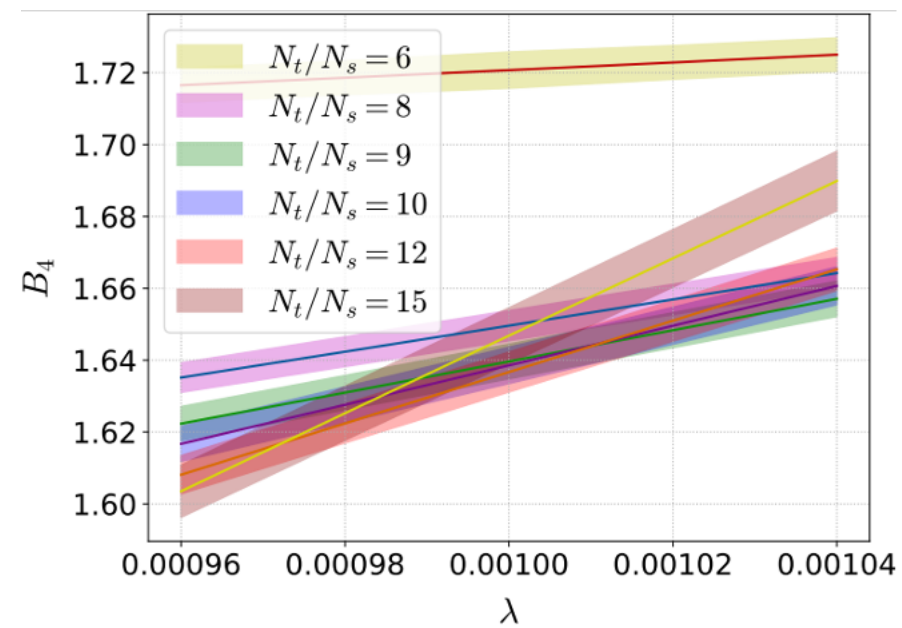
$N_t = 4$ vs. $N_t = 6$

Ashikawa+ (WHOT-QCD), *ongoing*

$N_t = 4$



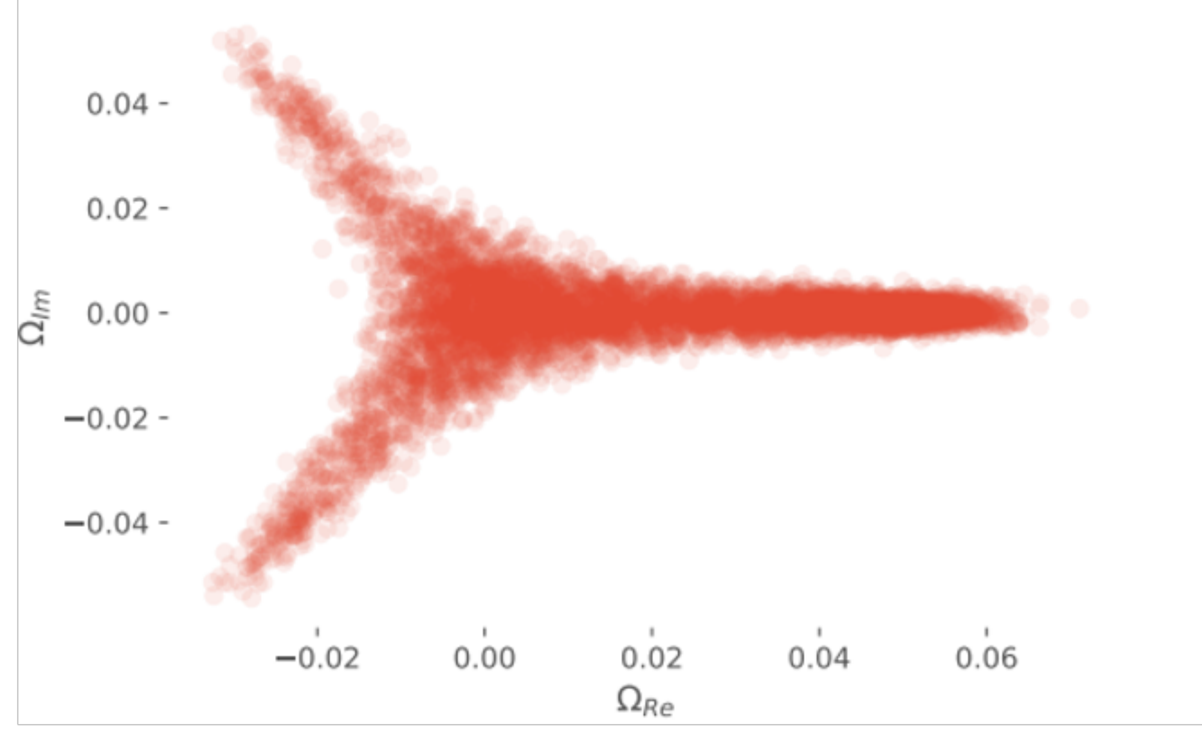
$N_t = 6$



- Violation of FSS larger on finer lattice => larger Ns/Nt = *LT* required

Origin of the violation:

- Contamination of remnant Z(3) distribution from $m_q = \infty$?
- Mixing with energy-like op. in the scaling ??



Conclusion & outlook

- **HPE provides us with a reliable and powerful way to study QCD with heavy quarks**
  - ☑ Convergent up to chiral limit + enable large Ns/Nt simul.'s + analytic in Nf
  - ☑ up to κ_c of Nt=4, Nf=2 : LO: ≥90% / NLO: ≥99% accurate
  - ☑ around κ_c of Nt=6, Nf=2 : NLO: ≥93% accurate

  Higher orders needed to remove remaining truncation error and for Nt≥8.

- **Effective method to incorporate high orders developed**
  => reduce simulation cost largely

- **At Nt=4, Ns/Nt≥9 needed for Z(2) FSS.**
  => NLO study of B_4^{Ω} : $\kappa_c = 0.0603(4)$ for Nf=2

- **At Nt=6, larger violation of FSS, require larger Ns/Nt**
  Preliminary with Ns/Nt≥12, $\kappa_c \sim 0.090$ including high orders.

- Nt=6 ongoing: more statistics.

- HPE powerful also at **finite densities**: in progress

We miss our best friend+collaborator

Yusuke Taniguchi

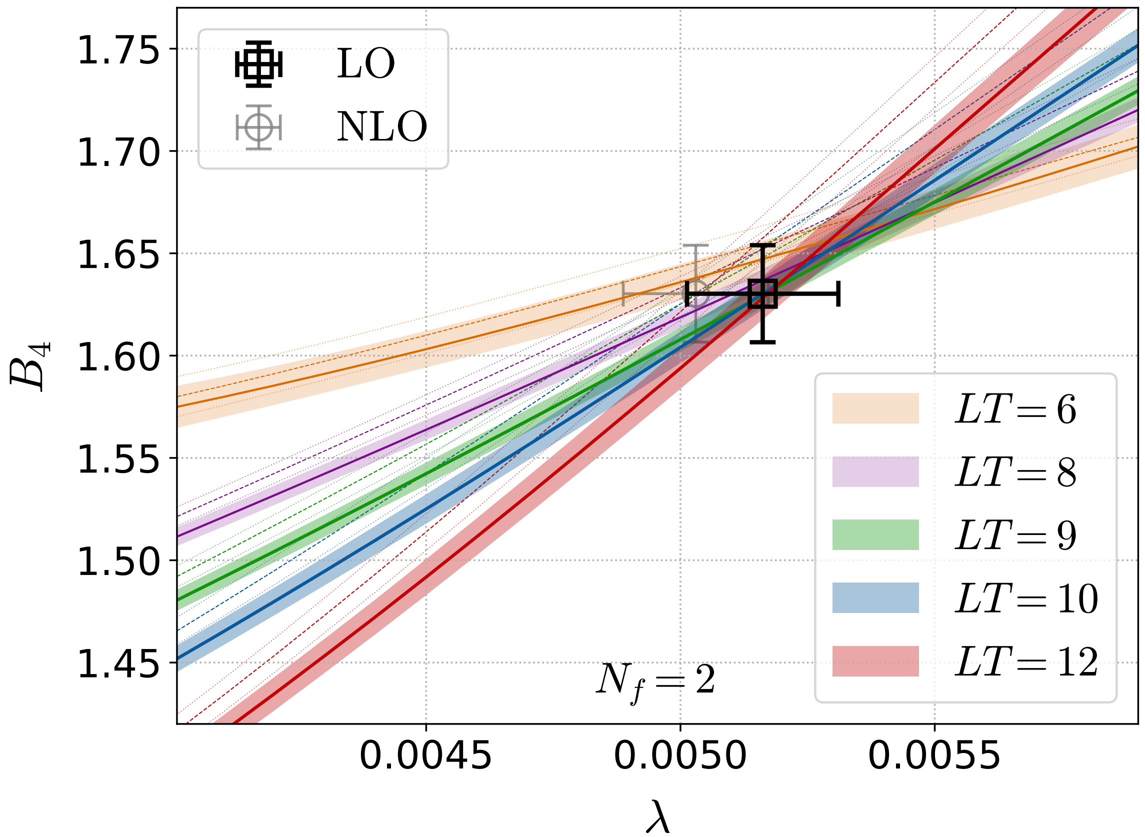
who passed away on July 22, 2022.

backup slides

Study on $N_t = 4$ lattices

Kiyohara, Kitazawa, Ejiri, KK (WHOT-QCD), *PRD* 104 (2021)

- Comparison with LO analysis => effects of NLO corrections



☆ LO ≈ NLO with Ns/Nt=LT ≥ 9

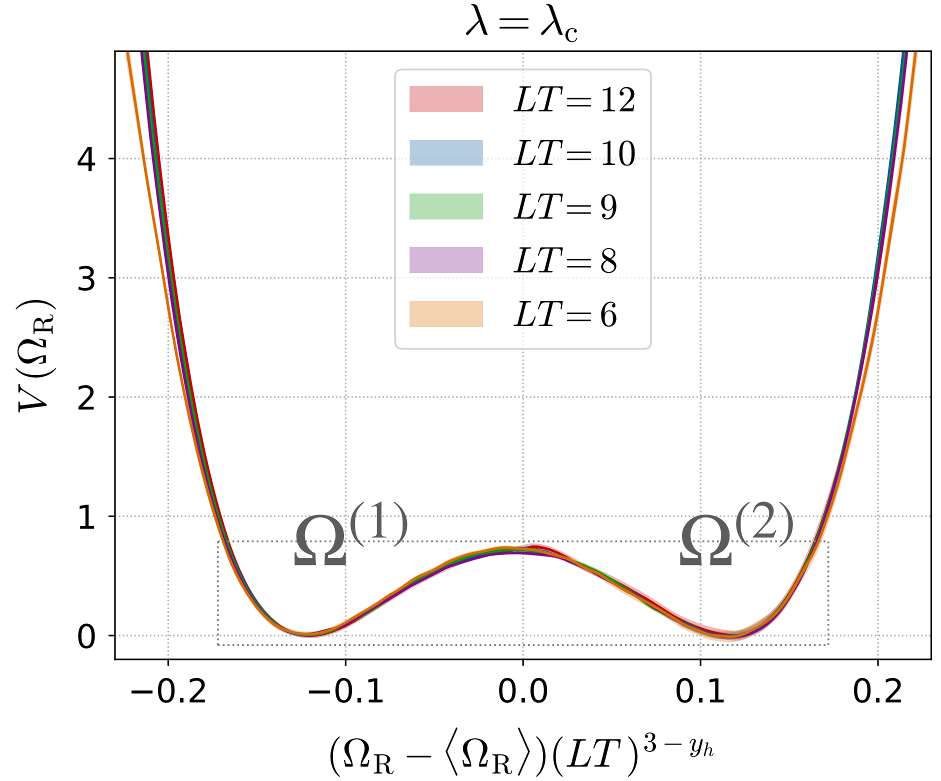
☆ Shift due to NLO is small ($\approx 2.6\%$), suggesting LO dominance around κ_c for Nt=4

=> previous Nt=4 LO results seems OK

backup slides

Kiyohara, Kitazawa, Ejiri, KK (WHOT-QCD), *PRD* 104 (2021)

$$V(\Omega_\mathrm{R};\lambda,LT) = -\ln p(\Omega_\mathrm{R})_{\lambda,LT},$$



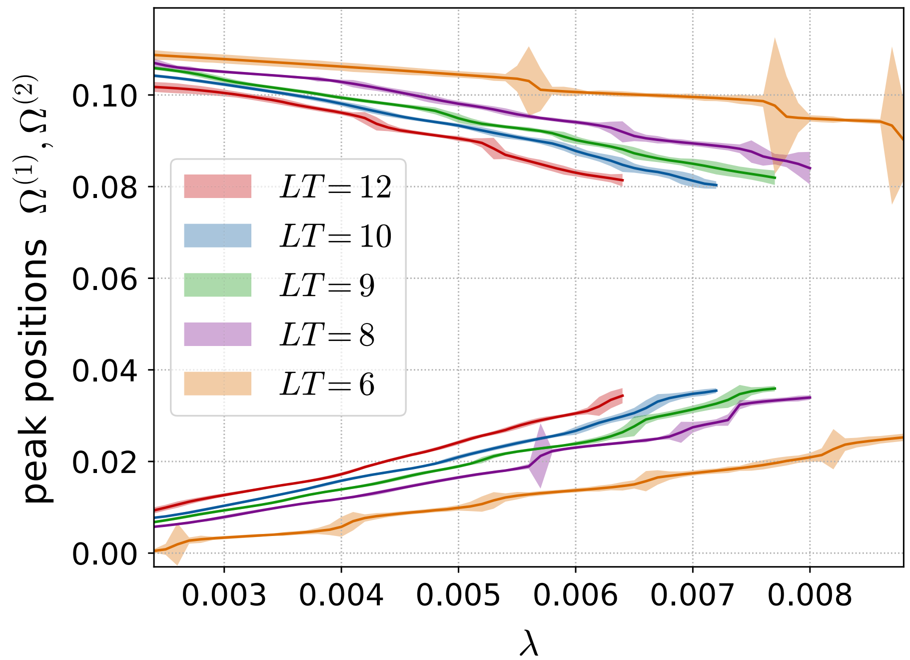
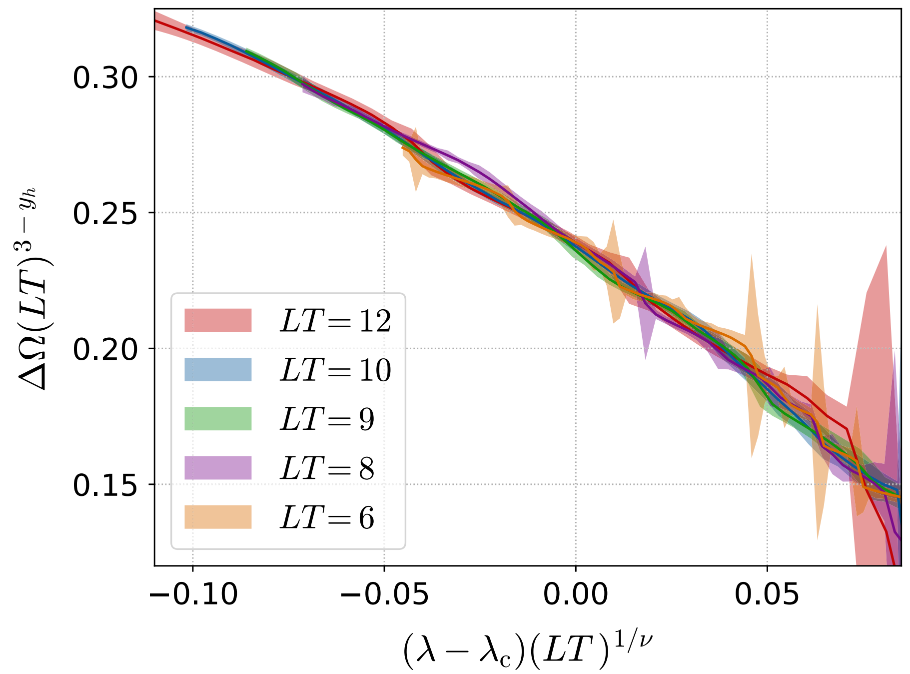


FIG. 13. Positions of peaks of the distribution function $p(\Omega_\mathrm{R})$ measured on the transition line.

$$\Delta\Omega = \Omega^{(2)} - \Omega^{(1)}. \tag{46}$$

According to Eq. (15), this quantity should behave around the *CP* as

$$\Delta\Omega(\lambda, LT) = (LT)^{y_h-3}\Delta\tilde{\Omega}((\lambda-\lambda_c)(LT)^{1/\nu}), \tag{47}$$



backup slides

Wakabayashi, Ejiri, KK, Kitazawa (WHOT-QCD), *PTEP* (2022)

$W^0(4)$	288	$W^0(20)$	$1.54422361 \times 10^{14}$	$W^0(36)$	$-5.58410362 \times 10^{27}$
$W^0(6)$	8448	$W^0(22)$	$2.83682900 \times 10^{15}$	$W^0(38)$	$-2.91018925 \times 10^{29}$
$W^0(8)$	245952	$W^0(24)$	$-2.40028584 \times 10^{16}$	$W^0(40)$	$-1.50223497 \times 10^{31}$
$W^0(10)$	7372800	$W^0(26)$	$-6.88836562 \times 10^{18}$	$W^0(42)$	$-7.71380102 \times 10^{32}$
$W^0(12)$	225232896	$W^0(28)$	$-5.41133954 \times 10^{20}$	$W^0(44)$	$-3.95168998 \times 10^{34}$
$W^0(14)$	6906175488	$W^0(30)$	$-3.39122203 \times 10^{22}$	$W^0(46)$	$-2.02386871 \times 10^{36}$
$W^0(16)$	208431502848	$W^0(32)$	$-1.93668514 \times 10^{24}$	$W^0(48)$	$-1.03783044 \times 10^{38}$
$W^0(18)$	$6.00259179 \times 10^{12}$	$W^0(34)$	$-1.05424635 \times 10^{26}$	$W^0(50)$	$-5.33468075 \times 10^{39}$

$L_1^0(4,4)$	48	$L_1^0(10,10)$	1228.8	$L_1^0(18,18)$	174762.67
$L_1^0(4,6)$	1728	$L_1^0(10,12)$	331776	$L_1^0(18,20)$	160432128
$L_1^0(4,8)$	45792	$L_1^0(10,14)$	52862976	$L_1^0(18,22)$	75497472000
$L_1^0(4,10)$	645120	$L_1^0(10,16)$	6258180096	$L_1^0(18,24)$	2.36626×10^{13}
$L_1^0(4,12)$	−26224128	$L_1^0(10,18)$	5.99330×10^{11}	$L_1^0(18,26)$	5.50232×10^{15}
$L_1^0(4,14)$	−3201067008	$L_1^0(10,20)$	4.87727×10^{13}	$L_1^0(18,28)$	1.01809×10^{18}
$L_1^0(4,16)$	-2.14087×10^{11}	$L_1^0(10,22)$	3.47446×10^{15}	$L_1^0(18,30)$	1.57315×10^{20}
$L_1^0(4,18)$	-1.19007×10^{13}	$L_1^0(10,24)$	2.20156×10^{17}	$L_1^0(20,20)$	629145.6
$L_1^0(4,20)$	-6.00757×10^{14}	$L_1^0(10,26)$	1.24531×10^{19}	$L_1^0(20,22)$	717225984
$L_1^0(4,22)$	-2.84486×10^{16}	$L_1^0(10,28)$	6.20798×10^{20}	$L_1^0(20,24)$	4.11140×10^{11}
$L_1^0(4,24)$	-1.28105×10^{18}	$L_1^0(10,30)$	2.59861×10^{22}	$L_1^0(20,26)$	1.54445×10^{14}
$L_1^0(4,26)$	-5.50874×10^{19}	$L_1^0(12,12)$	4096	$L_1^0(20,28)$	4.24543×10^{16}
$L_1^0(4,28)$	-2.25576×10^{21}	$L_1^0(12,14)$	1622016	$L_1^0(20,30)$	9.17892×10^{18}
$L_1^0(4,30)$	-8.69402×10^{22}	$L_1^0(12,16)$	360603648	$L_1^0(22,22)$	2287802.18
$L_1^0(6,6)$	128	$L_1^0(12,18)$	57416810496	$L_1^0(22,24)$	3170893824
$L_1^0(6,8)$	11520	$L_1^0(12,20)$	7.19497×10^{12}	$L_1^0(22,26)$	2.17478×10^{12}
$L_1^0(6,10)$	716544	$L_1^0(12,22)$	7.51820×10^{14}	$L_1^0(22,28)$	9.64167×10^{14}
$L_1^0(6,12)$	35891712	$L_1^0(12,24)$	6.80443×10^{16}	$L_1^0(22,30)$	3.09123×10^{17}
$L_1^0(6,14)$	1464910848	$L_1^0(12,26)$	5.46987×10^{18}	$L_1^0(24,24)$	8388608
$L_1^0(6,16)$	43817011200	$L_1^0(12,28)$	3.96931×10^{20}	$L_1^0(24,26)$	13891534848
$L_1^0(6,18)$	3.17933×10^{11}	$L_1^0(12,30)$	2.62442×10^{22}	$L_1^0(24,28)$	1.12307×10^{13}
$L_1^0(6,20)$	8.54676×10^{13}	$L_1^0(14,14)$	14043.43	$L_1^0(24,30)$	5.80075×10^{15}

backup slides

Wakabayashi, Ejiri, KK, Kitazawa (WHOT-QCD), *PTEP* (2022)

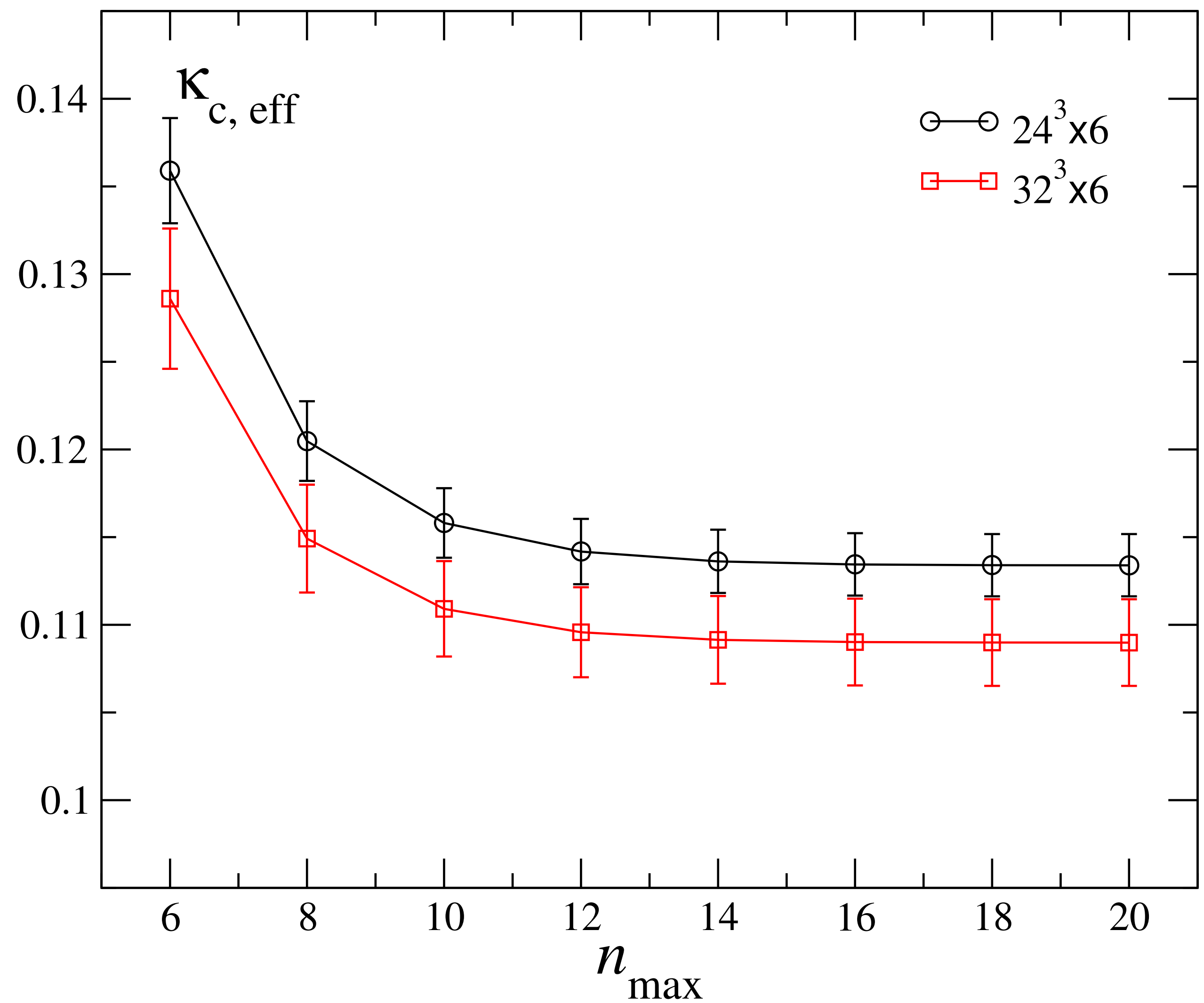


Fig. 11. Effective critical point $\kappa_{c,\,eff}$ in two-flavor QCD for $N_t = 6$ as a function of n_{max}. The black circle and red square symbols are for $\kappa_{c,\,LO}$ obtained on a $24^3 \times 6$ and a $32^3 \times 6$ lattice, respectively.

backup slides

Wakabayashi, Ejiri, KK, Kitazawa (WHOT-QCD), *PTEP* (2022)

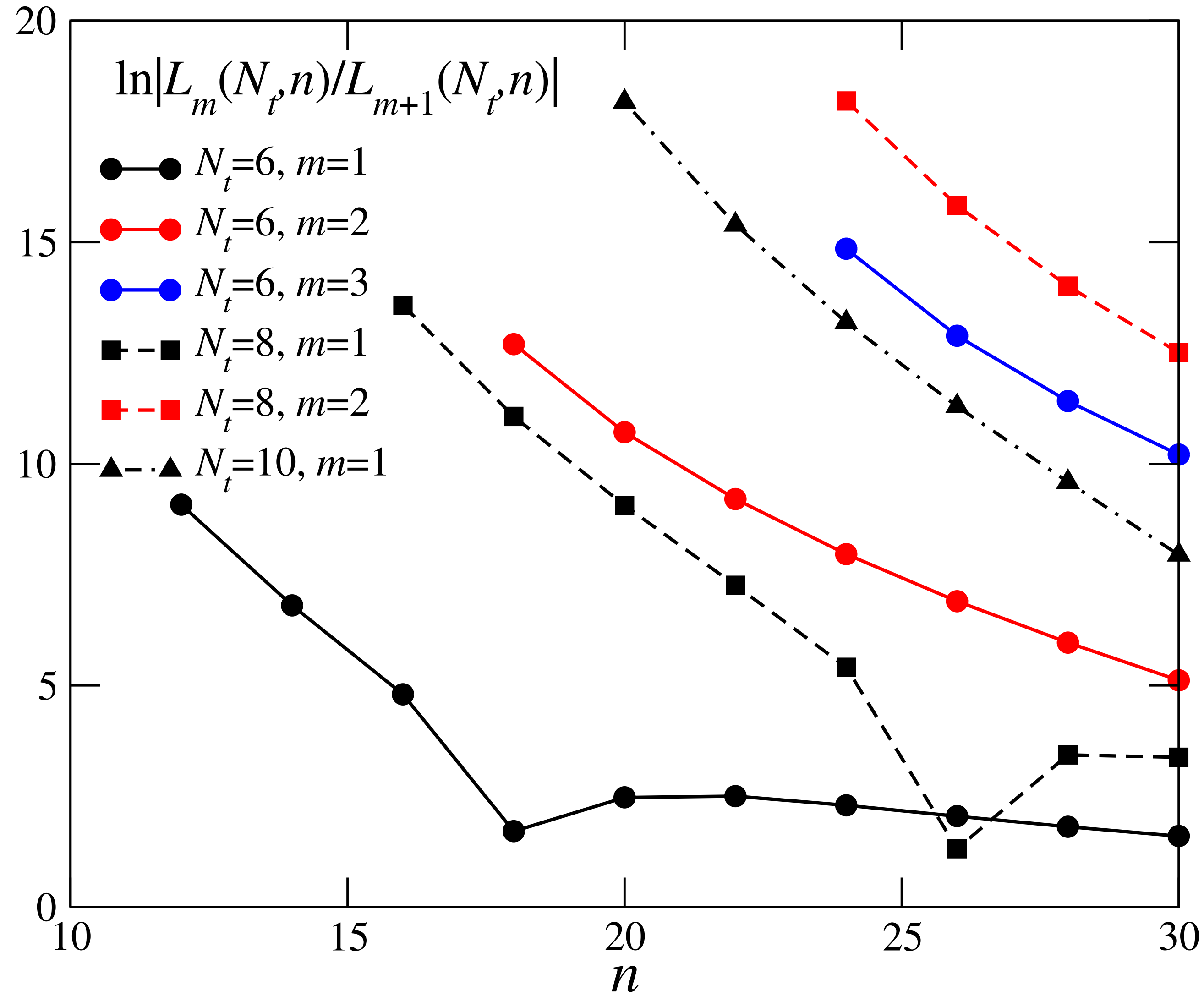


Fig. 14. Upper bound of μ/T such that higher-m terms are small, as given in Eq. (71).

$$\frac{\mu}{T} < \ln\left|\frac{L^0_m(N_t, n)}{L^0_{m+1}(N_t, n)}\right|$$