“Tensor renormalization group study of (1+1)-dimensional O(3) nonlinear sigma model”

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The 1st TCHoU member meeting in 2024, July 2, 2024

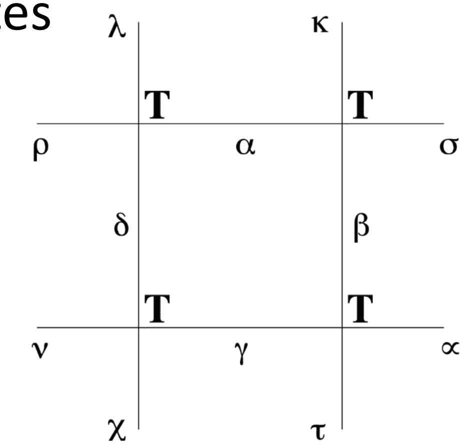
Plan of talk

- Introduction to Tensor Renormalization Group(TRG)
- Application of TRG to Quantum Field Theories(QFTs)
- Entanglement Entropy(EE) of (1+1)d O(3) NLSM at $\mu = 0$
- Quantum Phase Transition of (1+1)d O(3) NLSM at $\mu \neq 0$
- Summary and Outlook

Tensor Renormalization Group (TRG)

Levin-Nave
PRL99(2007)120601

Explain the algorithm with 2D Ising model with N sites



Hamiltonian $H = \sum_{\langle i,j \rangle} s_i s_j \qquad s_i \pm 1$

Partition Function $Z = \sum_{\{s_i\}} \exp\left(-\beta H\right)$

$$= \sum_{\alpha,\beta,\gamma,\delta,\cdots=1}^{2} T_{\alpha,\lambda,\rho,\delta} T_{\sigma,\kappa,\alpha,\beta} T_{\mu,\beta,\gamma,\tau} T_{\gamma,\delta,\nu,\chi} \cdots$$

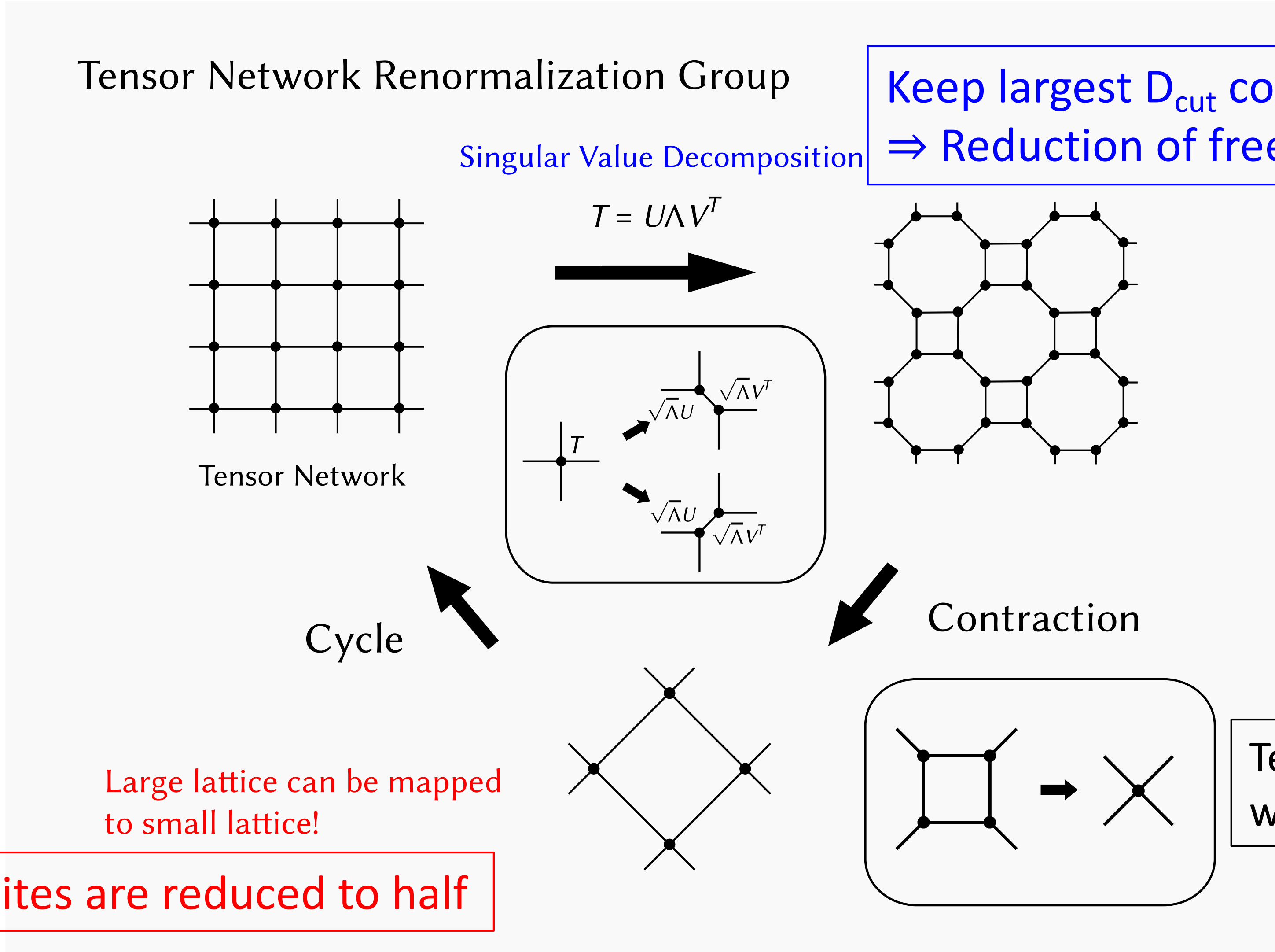
Tensor Network representation

Details of model are specified in initial tensor
The algorithmic procedure is independent of models

Of course, direct contraction is impossible for large N even with current fastest supercomputer
⇒ How to evaluate the partition function?

Schematic View of TRG Algorithm

1. Singular Value Decomposition of local tensor T
2. Contraction of old tensor indices (coarse-graining)
3. Repeat the iteration



Keep largest D_{cut} components
⇒ Reduction of freedom

Large lattice can be mapped
to small lattice!

#sites are reduced to half

Tensors
w/ new indices

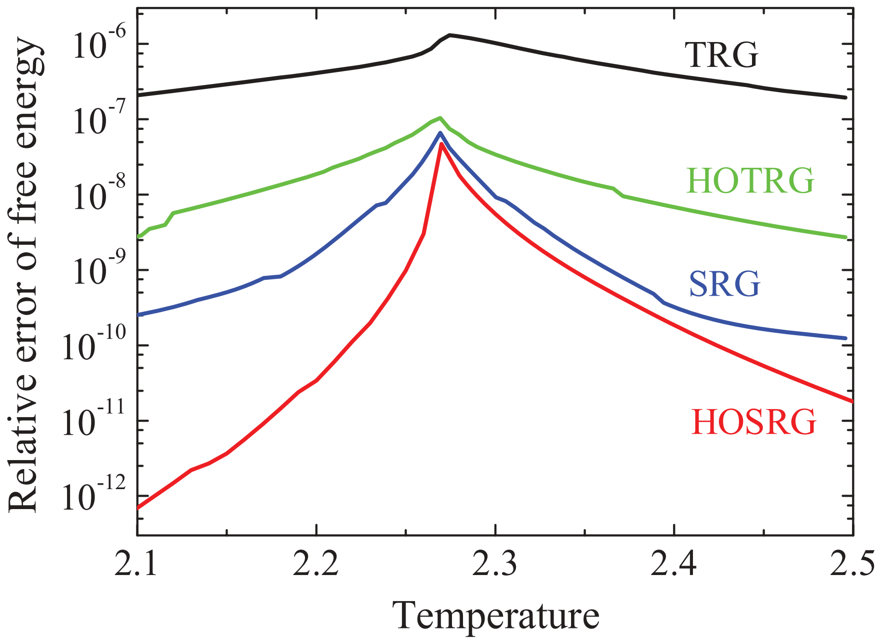
Numerical test for 2d Ising Model

The key element in the algorithm is low-rank approximation by SVD

$$T_{i,j,k,l} \simeq \sum_{m=1}^{\mathrm{D_{cut}}} U_{(i,j),m} \sigma_m V_{m,(k,l)}$$

Truncation error is controlled by the parameter $\mathrm{D_{cut}}$

Free energy on and off the transition point, lattice size=$2^{30\sim50}$, $\mathrm{D_{cut}}$=24



Xie et al.
PRB86(2012)045139

Comparison with analytic results

Relative error of free energy : $\leq 10^{-6}$

TRG vs Monte Carlo

Monte Carlo	antithetical principles	TRG
stochastic	⟷	deterministic

Advantages of TRG

- Free from sign problem/complex action problem in MC method

$$Z = \int \mathcal{D}\phi \ \exp(-S_{\rm Re}[\phi] + iS_{\rm Im}[\phi])$$

- Computational cost for L^D system size $\propto D \times \log(L)$
- Direct manipulation of Grassmann numbers
- Direct evaluation of partition function Z (density matrix ρ) itself

⇓

Applications in particle physics :

Finite density QCD, QFTs w/ θ-term, Lattice SUSY etc.

Also, in condensed matter physics

Hubbard model (Mott transition, High Tc superconductivity) etc.

TRG Approaches to QFTs (1)

w/ sign problem

2d models

- CP(1) model w/ θ-term : Kawauchi-Takeda, EPJWoC175(2018)11015
- O(3) NLSM : Luo-YK, JHEP03(2024)020
- Real ϕ^4 theory :
  - Shimizu, Mod.Phys.Lett.A27(2012)1250035
  - Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP05(2019)184
- Complex ϕ^4 theory at finite density :
  - Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP02(2020)161
- U(1) gauge theory w/ θ-term :
  - YK-Yoshimura, JHEP04(2020)089
- Schwinger(2d QED), Schwinger w/ θ-term :
  - Shimizu-YK, PRD90(2014)014508, 074503, PRD97(2018)034502
- Gross-Neveu model at finite density :
  - Takeda-Yoshimura, PTEP2015(2015)043B01
- N=1 Wess-Zumino model (SUSY) :
  - Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP03(2018)141

Application to models w/ sign problem,
Development of calculational methods for scalar, fermion and gauge fields

TRG Approaches to QFTs (2)

w/ sign problem

3d models

Z_2 gauge Higgs model at finite density : Akiyama-YK,JHEP05(2022)102
Real ϕ^4 theory : Akiyama-YK-Yoshimura, PRD104(2021)034507
Z_2gauge theory at finite temperature : YK-Yoshimura, JHEP08(2019)023

4d models

Ising model : Akiyama-YK-Yamashita-Yoshimura, PRD100(2019)054510
Complex ϕ^4 theory at finite density :
Akiyama-Kadoh-YK-Yamashita-Yoshimura, JHEP09(2020)177
NJL model at finite density :
Akiyama-YK-Yamashita-Yoshimura, JHEP01(2021)121
Real ϕ^4 theory : Akiyama-YK-Yoshimura, PRD104(2021)034507
Z_2 gauge Higgs model at finite density : Akiyama-YK, JHEP05(2022)102
Z_3 gauge Higgs model at finite density : Akiyama-YK, JHEP10(2023)077

⇒ Research target is shifting from 2d models to 4d ones

TRG Approaches to QFTs (3)

w/ sign problem

<u>Condensed matter physics</u>

Similarity btw Hubbard models and NJL ones

Action consisting of hopping terms and 4-fermi interaction term

$$S=\sum_{n\in\Lambda_{d+1}}\epsilon\left\{\bar{\psi}(n)\left(\frac{\psi(n+\hat{\tau})-\psi(n)}{\epsilon}\right)-t\sum_{\sigma=1}^{d}(\bar{\psi}(n+\hat{\sigma})\psi(n)+\bar{\psi}(n)\psi(n+\hat{\sigma}))+\frac{U}{2}(\bar{\psi}(n)\psi(n))^2-\mu\bar{\psi}(n)\psi(n)\right\}$$

First principle calculation at finite density

(1+1)d Hubbard model : Akiyama-YK, PRD104(2021)014504

(2+1)d Hubbard model : Akiyama-YK-Yamashita, PTEP2022(2022)023I01

<u>In this talk we focus on (1+1)d O(3) NLSM</u>

Entanglement Entropy (EE) at $\mu=0$

Direct evaluation of partition function Z (density matrix ρ) itself

Quantum phase transition at $\mu\neq 0$

Free from sign problem/complex action problem

Determination of dynamical critical exponent *z*

EE of (1+1)d O(3) NLSM at $\mu = 0$

Luo-YK, JHEP03(2024)020

(1+1)d lattice O(3) NLSM (asymptotic free)

$$Z = \int \mathcal{D}[\boldsymbol{s}] e^{-S}$$

$$S = -\beta \sum_{n \in \Lambda_{1+1}, \nu} \boldsymbol{s}(n) \cdot \boldsymbol{s}(n + \hat{\nu})$$

$$\boldsymbol{s}^T(\Omega) = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$$
$$\Omega = (\theta, \phi) \quad , \ \theta \in (0, \pi], \ \phi \in (0, 2\pi].$$

(θ, ϕ) is discretized w/ Gauss-Legendre quadrature → TN representation

Entanglement Entropy(EE)

Whole system (V=2L × Nt) is divided to subsystems A, B (V_A,V_B=L × Nt)

von Neumann

$$S_A = -\mathrm{Tr}_A \rho_A \log(\rho_A)$$

$$\rho_A = \frac{1}{Z} \mathrm{Tr}_B [T \cdots T]$$

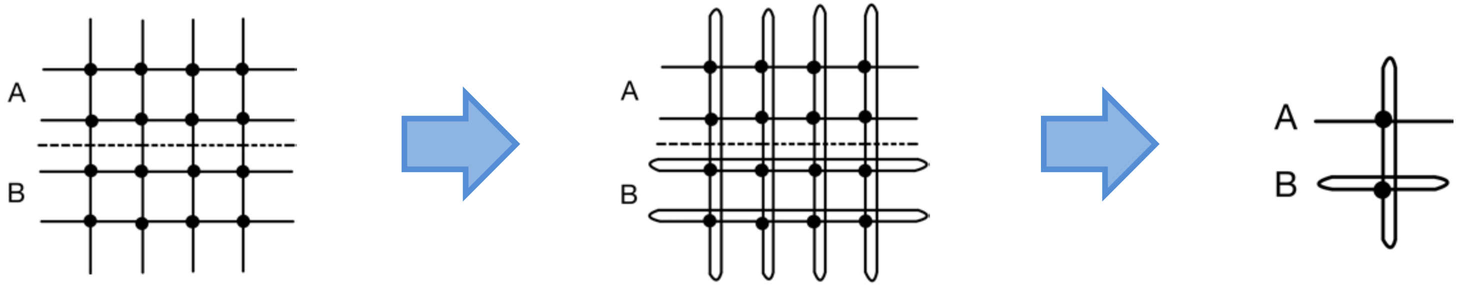
Rényi(n-th)

$$S_A^{(n)} = \frac{\ln \mathrm{Tr}_A \rho_A^n}{1 - n}$$

Calculation of EE

Luo-YK, JHEP03(2024)020

von Neumann EE

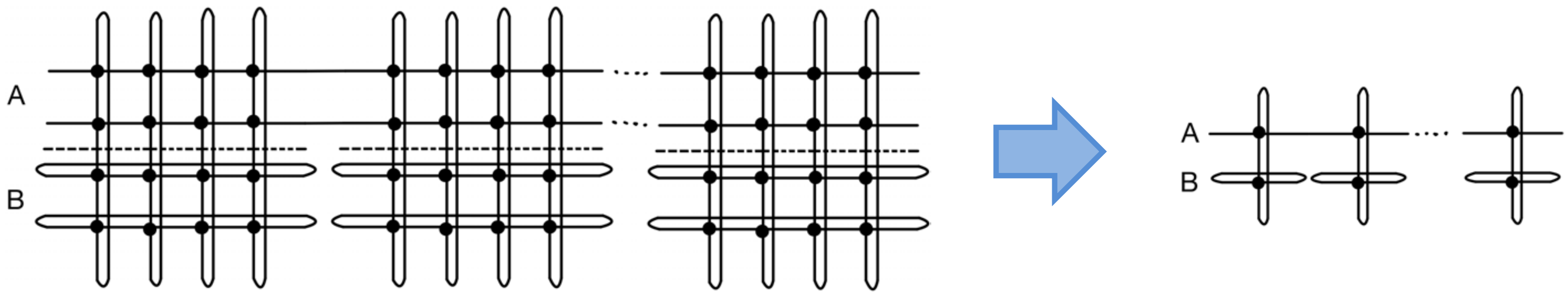


Divided into subsystems A and B

Trace out in terms of subsystem B (Tr_B)

Coarse-graining w/ HOTRG

n-th Rényi EE

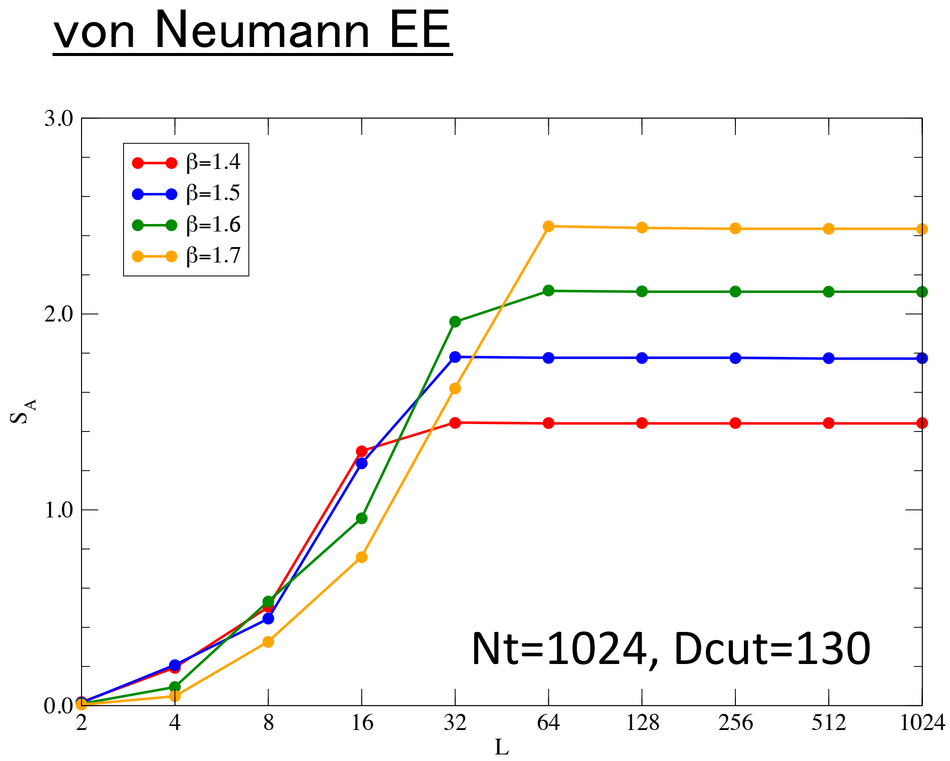


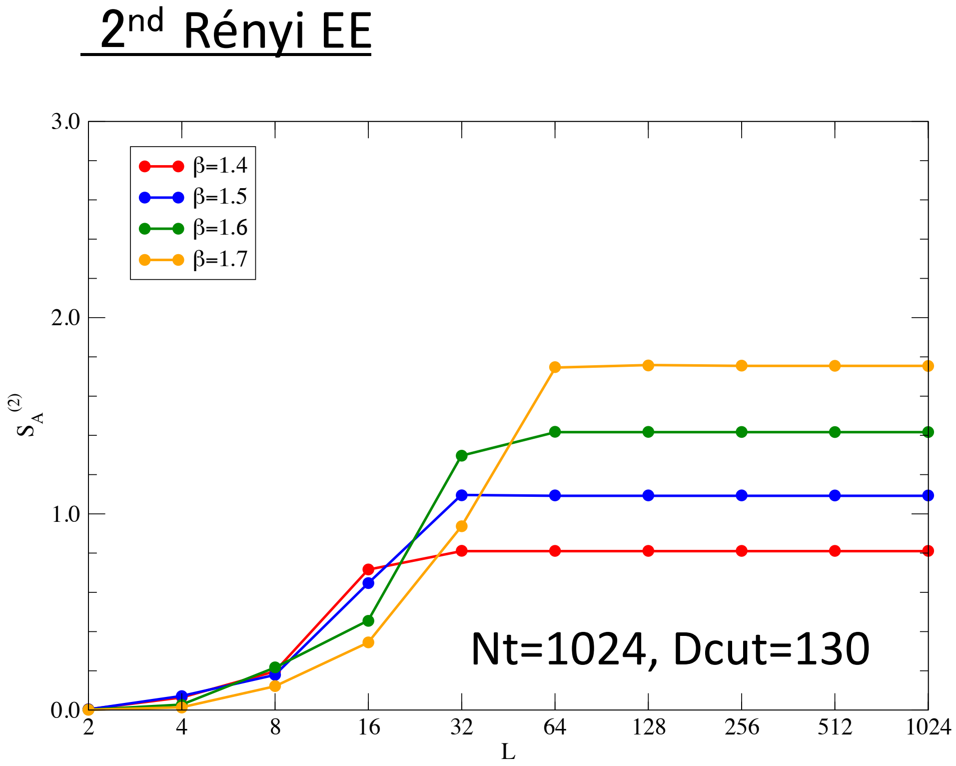
Tr_B for subsystem B on each sheet of n- times copied system

Coarse-graining w/ HOTRG

Spatial size (L) dependence of EE

Luo-YK, JHEP03(2024)020





$S_A \sim \frac{c}{3}\ln(\xi)$ (c: central charge, ξ: correlation length) $S_A^{(2)} \sim \frac{c}{3}(1+\frac{1}{2})\ln(\xi)$

Convergence at $L \gg \xi$ is confirmed

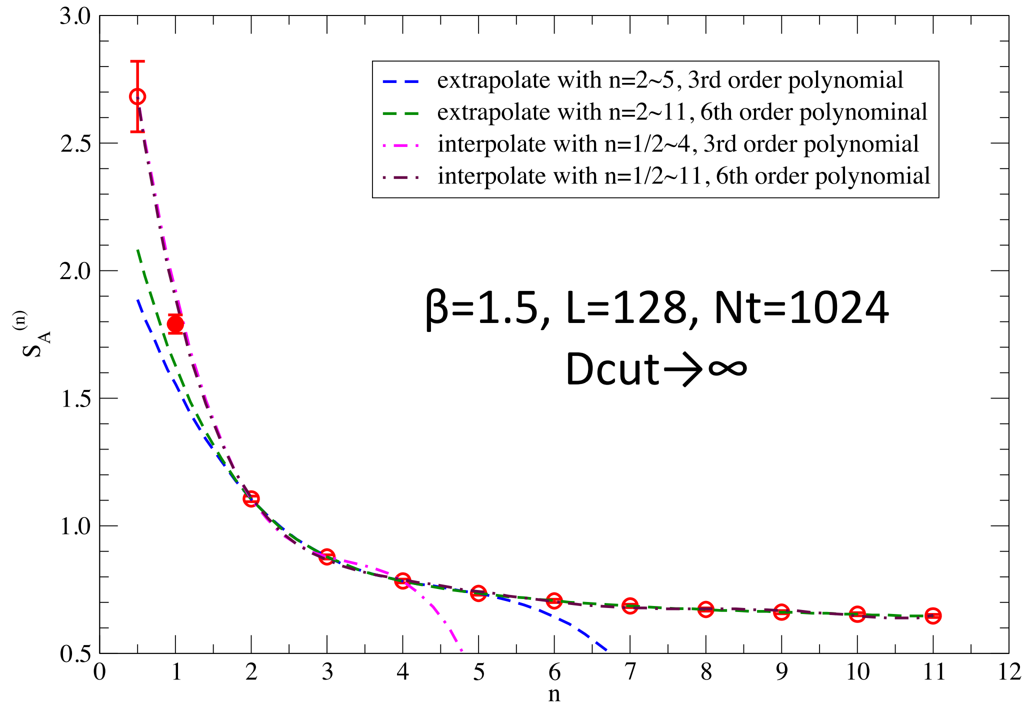
β	1.4	1.5	1.6	1.7
ξ	6.90(1)	11.09(2)	19.07(6)	34.57(7)

Wolff, NPB334(1990)581

von Neumann vs Rényi

Luo-YK, JHEP03(2024)020

Comparison btw von Neumann EE(n=1) and Rényi EE(n≠1)



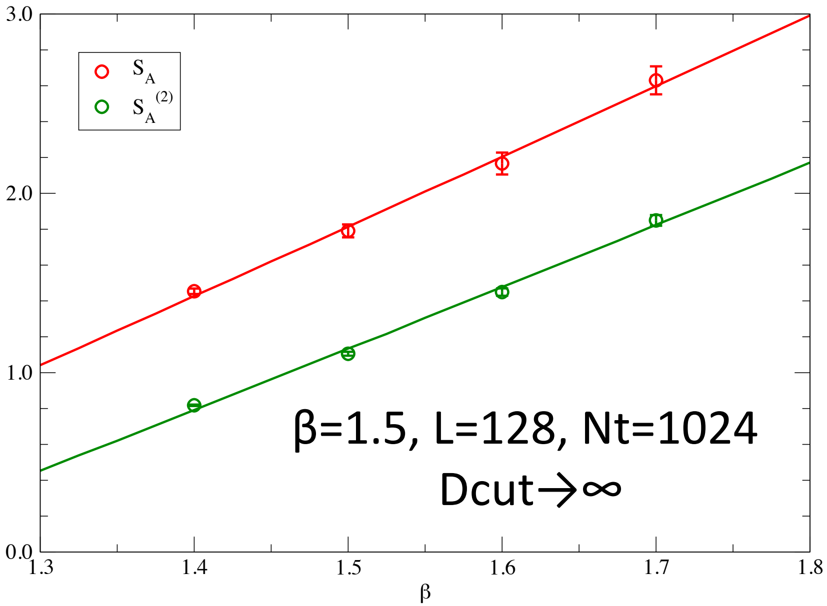
$S_A^{(2)}$ is not a good approximation of von Neumann EE(n=1)

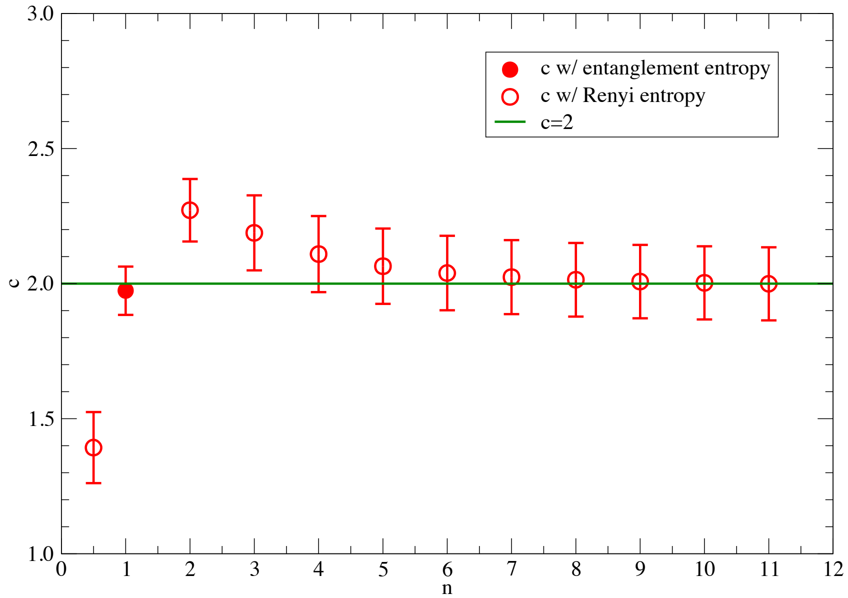
Difficult to extrapolate $S_A^{(n)}$ ($n \geq 2$) to n=1 at high precision

Reliable interpolation to n=1 using $S_A^{(1/2)}$ and $S_A^{(n)}$ ($n \geq 2$)

Central Charge

Luo-YK, JHEP03(2024)020





mass gap(two loop): $m = \frac{8}{e}\Lambda_{\overline{\mathrm{MS}}} = 64\Lambda_L = \frac{128\pi}{a}\beta \exp\left(-2\pi\beta\right)$

von Neumann EE: $S_A = \frac{c}{3}\left(2\pi\beta - \ln\beta\right) + \text{const.}$

n-th Rényi EE: $S_A^{(n)} = \frac{c}{6}\left(1+\frac{1}{n}\right)\left(2\pi\beta - \ln\beta\right) + \text{const.}$

central charge is determined from β dependence

c=1.97(9) (von Neumann EE)

consistent btw different methods

c=2.04(4) by Matrix Product State (MPS)

Bruckmann+, PRD99(2019)074501

c∼2 by finite size spectrum w/ TNR

Ueda+, PRE106(2022)014104

Quantum phase transition at $\mu \neq 0$

Luo-YK, arXiv:2406.08865

(1+1)d lattice O(3) NLSM at $\mu\neq 0$

$$Z = \int \mathcal{D}[\boldsymbol{s}] e^{-S}$$

$$S = -\beta \sum_{n\in\Lambda_{1+1},\nu} \sum_{\lambda,\gamma=1}^{3} s_\lambda(\Omega_n) D_{\lambda\gamma}(\mu,\hat{\nu}) s_\gamma(\Omega_{n+\hat{\nu}})$$

$$D(\mu,\hat{\nu}) = \begin{pmatrix} 1 & & \\ & \cosh(\delta_{2,\nu}\mu) & -i\sinh(\delta_{2,\nu}\mu) \\ & i\sinh(\delta_{2,\nu}\mu) & \cosh(\delta_{2,\nu}\mu) \end{pmatrix}$$ ⇒ complex action

$$\boldsymbol{s}^T(\Omega) = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$$
$$\Omega = (\theta,\phi) \quad , \ \theta \in (0,\pi], \ \phi \in (0,2\pi].$$

(θ,ϕ) is discretized w/ Gauss-Legendre quadrature → TN representation

Symmetry btw spatial and temporal directions is broken by μ

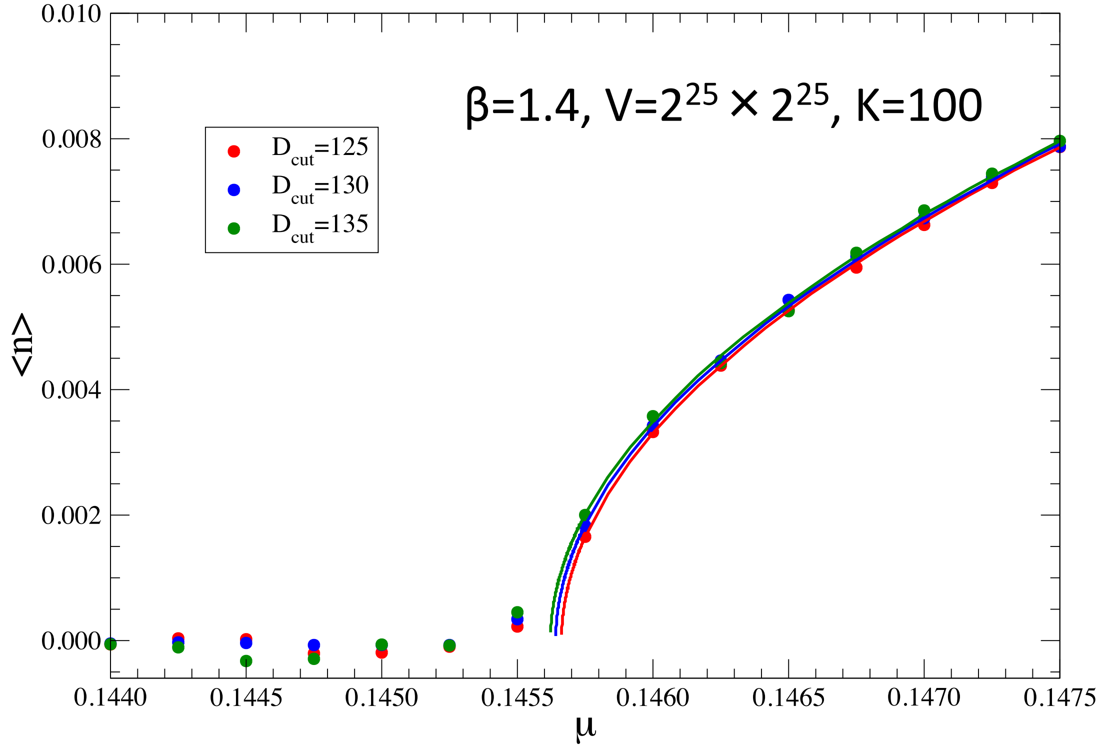
⇒ Spatial correlation length (ξ) $\neq$ Temporal correlation length (ξ_t)

$\xi_t = \xi^z$ with z dynamical critical exponent

Number Density

Luo-YK, arXiv:2406.08865

number density: $\langle n\rangle = \frac{\partial}{\partial\mu} f \approx \frac{-1}{LN_t}\frac{\ln Z(\mu+\Delta\mu)-\ln Z(\mu-\Delta\mu)}{2\Delta\mu}$



Simultaneous fit with $\langle n\rangle(\mu, D_{\rm cut}) = A_n \cdot \{\mu - (\mu_{\rm c} + B_n/D_{\rm cut})\}^{\nu}$

$$\Rightarrow \nu = 0.512(15), \mu_c = 0.14512(11)$$

μ_c is consistent with mass gap m=0.1449(2) at $\mu = 0$ Wolff, NPB334(1990)581

Temporal Correlation Length (1)

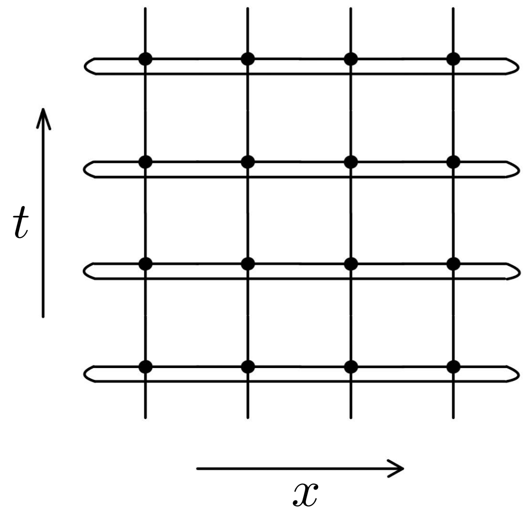
Luo-YK, arXiv:2406.08865

ξ_t is obtained from the eigenvalues of density matrix

$$\xi_t = \frac{N_t}{\ln\left(\frac{\lambda_0}{\lambda_1}\right)}$$

λ_0 and λ_1 are the largest and second largest eigenvalues

ex. density matrix on 4 × 4 lattice

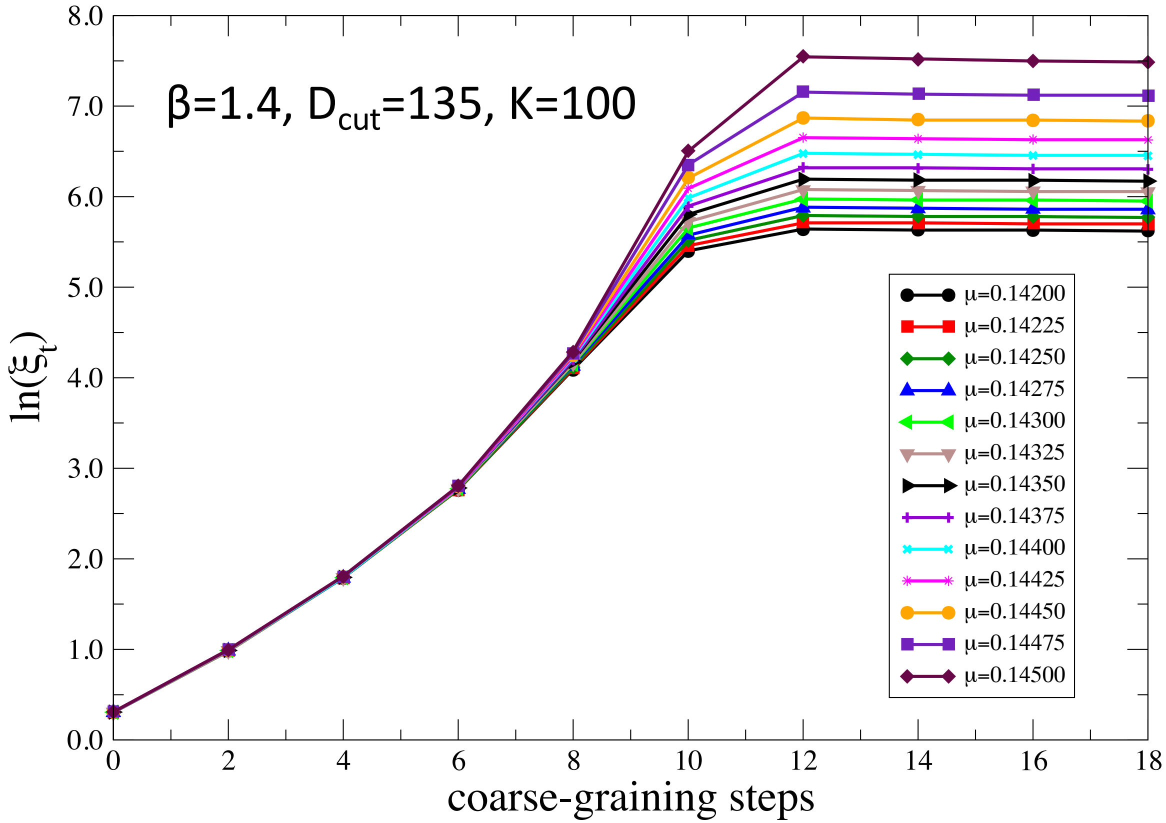


Eigenvalues are calculated on reduced single tensor obtained by HOTRG

Temporal Correlation Length (2)

Luo-YK, arXiv:2406.08865

ξ_t as a function of coarse-graining steps near μ_c

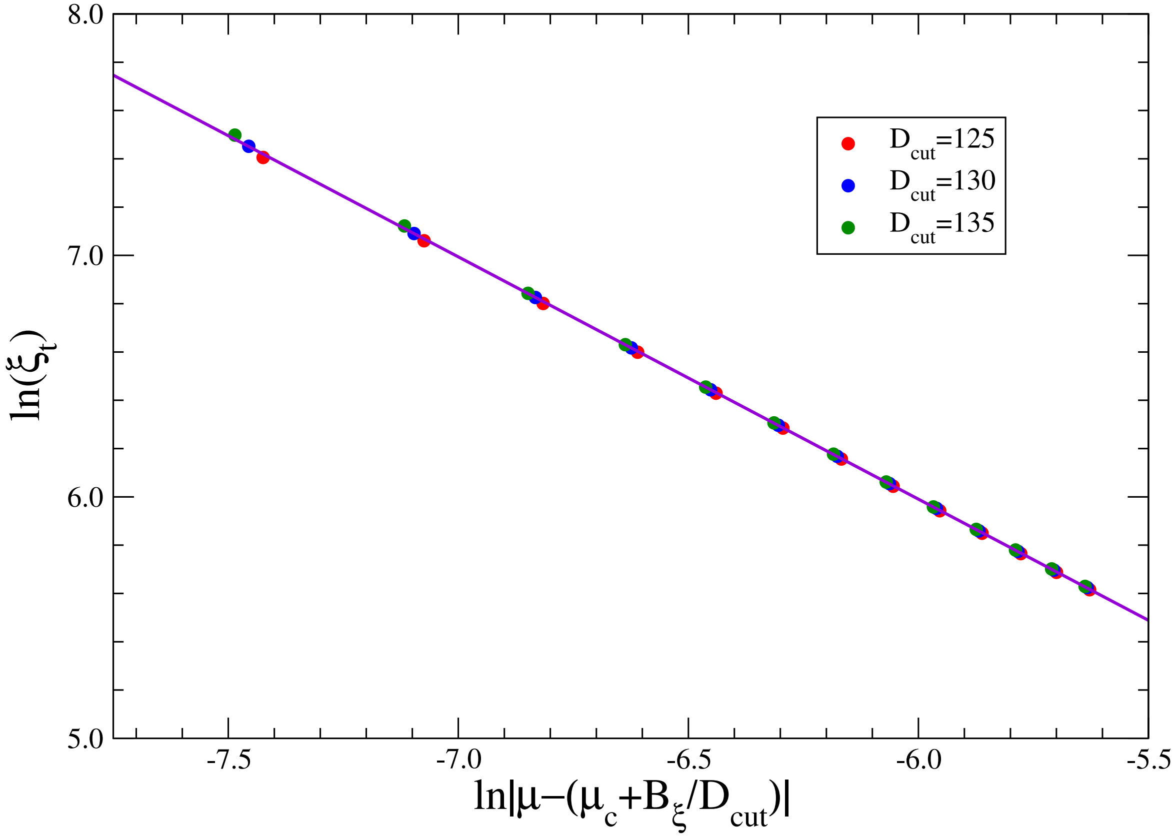


Plateau behaviors are observed for $L > \xi$ at sufficiently low temperature

Temporal Correlation Length (3)

Luo-YK, arXiv:2406.08865

μ dependence of ξ_t near μ_c



Simultaneous fit with $\ln \xi_t(\mu, D_{\text{cut}}) = A_\xi + \alpha \ln |\mu - (\mu_c + B_\xi / D_{\text{cut}})|$

$$\Rightarrow \alpha = z\nu = 1.003(5), z = 1.96(6)$$

The first successful calculation of dynamical critical exponent with TRG

Summary and Outlook

<u>Advantages of TRG</u>

- Free from sign problem/complex action problem in MC method

$$Z = \int \mathcal{D}\phi \ \exp(-S_{\rm Re}[\phi] + iS_{\rm Im}[\phi])$$

- Computational cost for L^D system size $\propto D \times \log(L)$
- Direct manipulation of Grassmann numbers
- Direct evaluation of partition function Z itself

⇓

Applications in particle physics :

Finite density QCD, QFTs w/ θ-term, Lattice SUSY etc.

Also, in condensed matter physics

Hubbard model (Mott transition, High Tc superconductivity) etc.

Current status : Calculations of 4d theories (scalar, fermion, gauge) are possible

⇒ Extension to Abelian (U(1)) and non-Abelian groups (SU(2), SU(3))

Non-perturbative study of Entanglement Entropy