2025/12/16
TCHoU成果報告会

Lee-Yangゼロ比法によるスピン系と重クォークQCDの臨界点探査

Search for the critical point in spin systems and heavy-quark-QCD using the Lee-Yang Zero Ratio method

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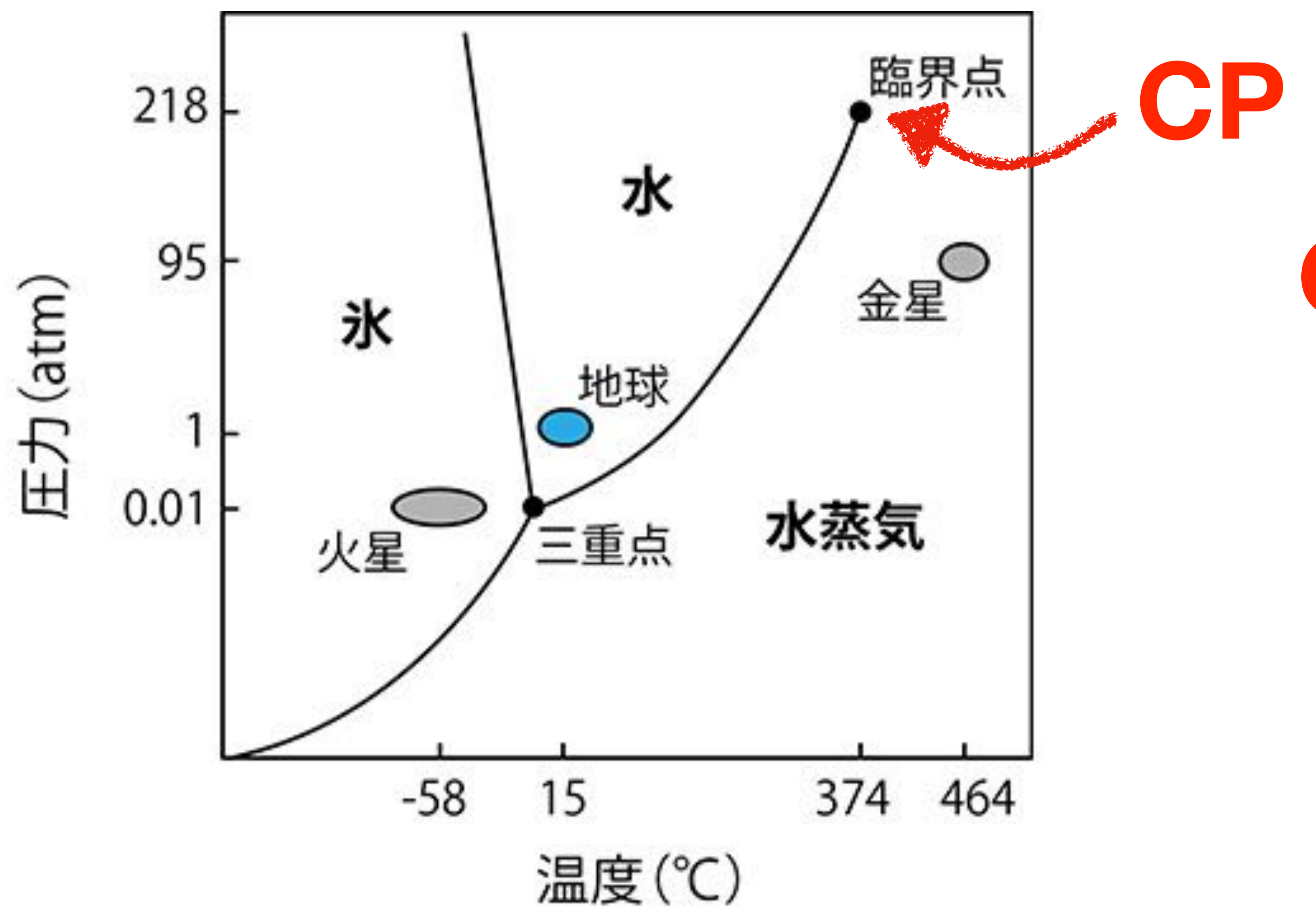
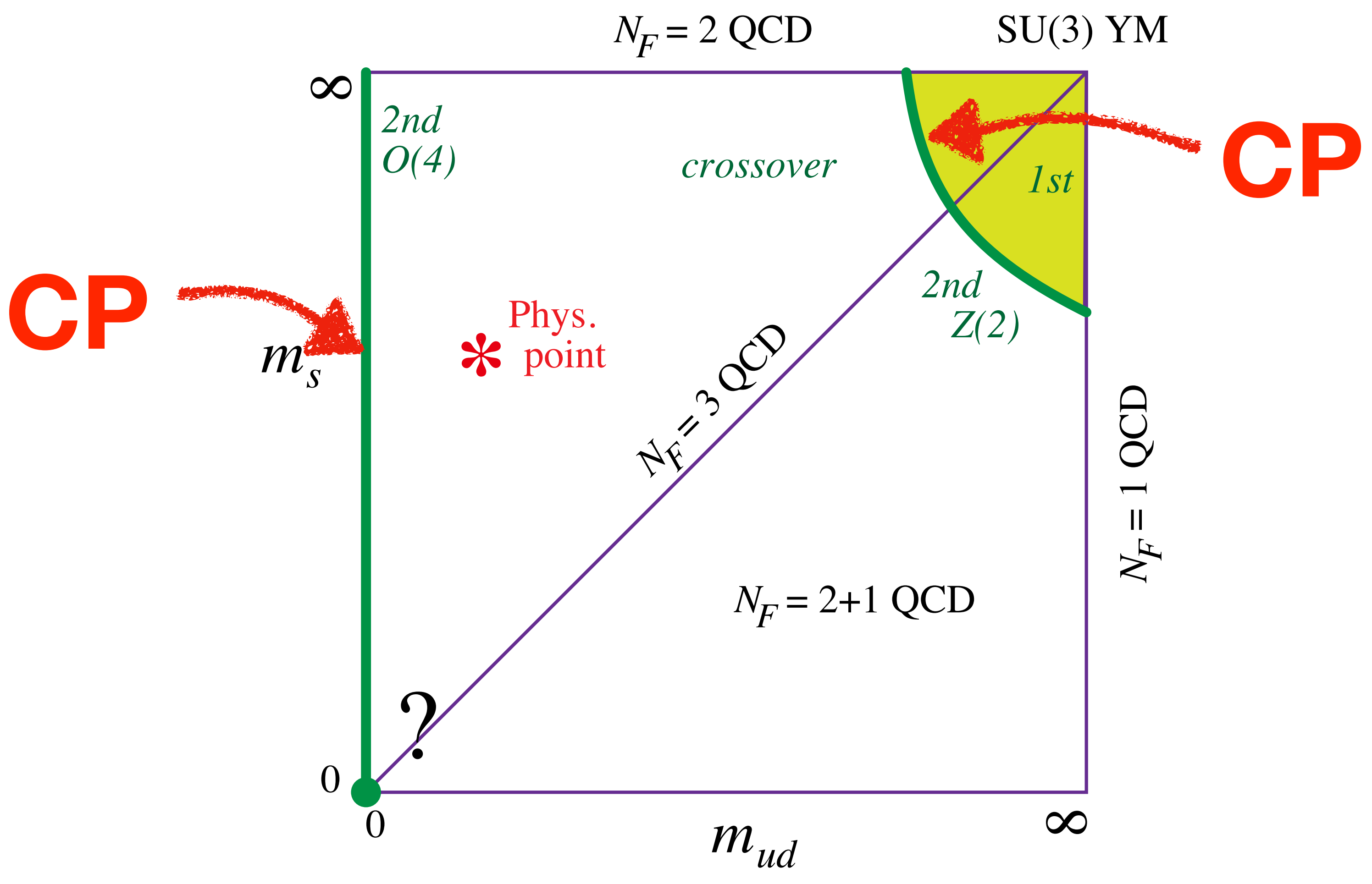
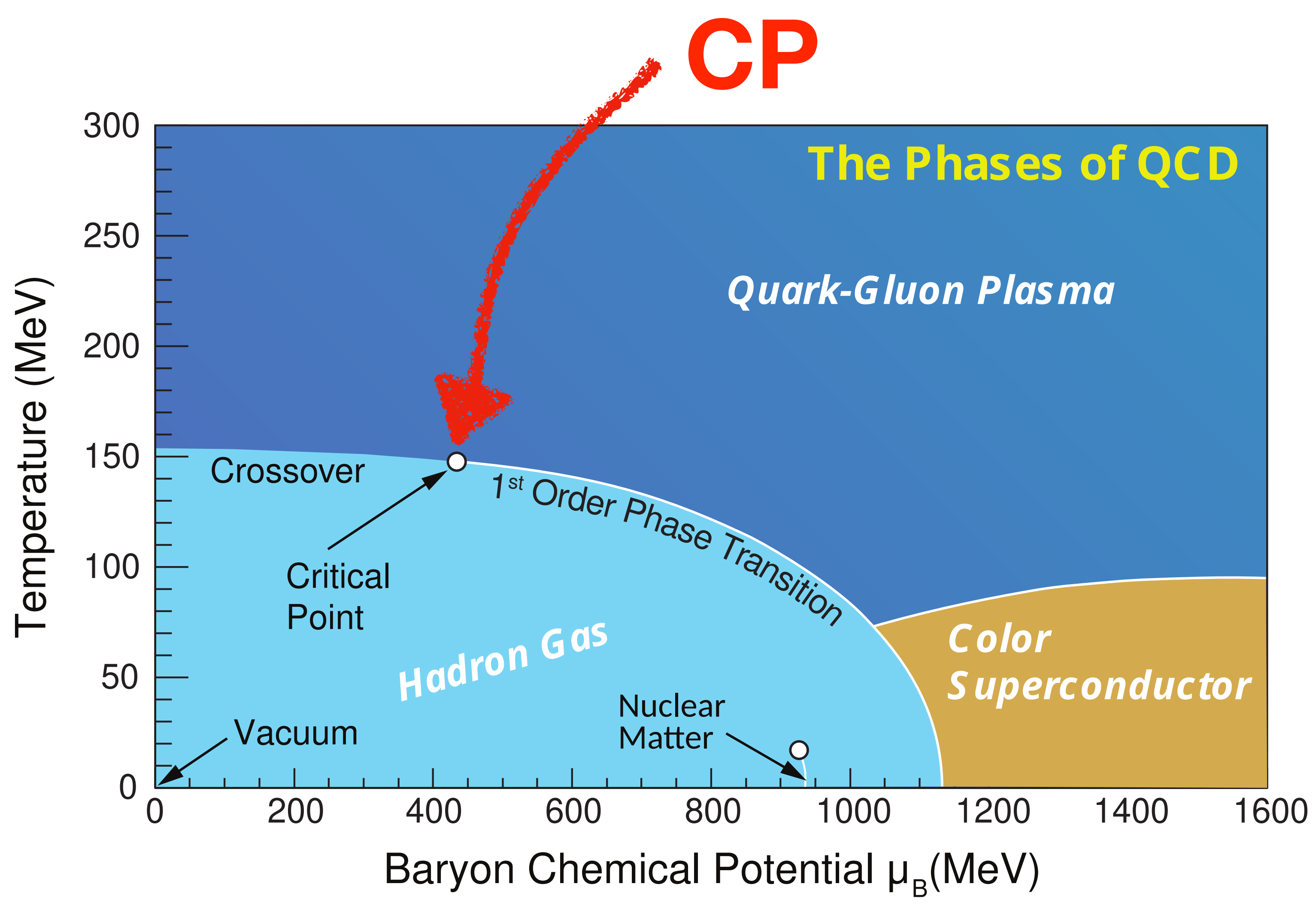
- **T. Wada, M. Kitazawa, K. Kanaya, *Phys. Rev. Lett. 134, 162302 (2025)***
- **T. Wada, M. Kitazawa, K. Kanaya, *arXiv:2508.20422 → JPSJ (2026)***
- **T. Wada, M. Kitazawa, K. Kanaya, *on-going***

Critical Points

important landmark in investigating thermodynamic properties







This report: FSS of Lee-Yang Zeros ⇒ ratios of LYZs (LYZRs) similar to the Binder cumulant

⇒ a new general tool to determine CPs

- tests in spin systems ⇒ LYZR is as powerful as the Binder cumlant

⇒ LYZR superior in suppressing various corrections

- application to heavy-quark QCD (status report)

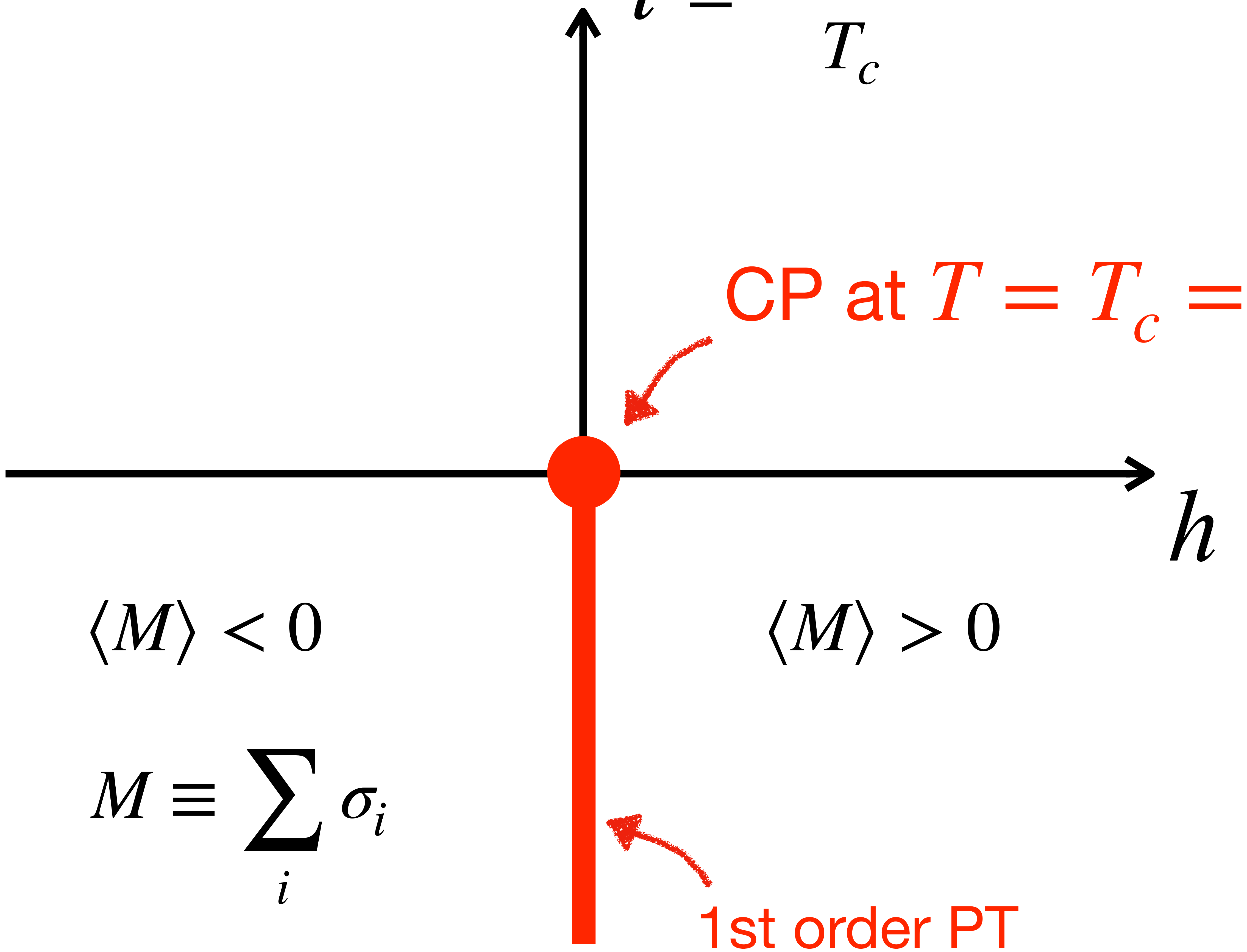
3d-Ising

$\sigma_i = \pm 1$

$$Z = \sum_{\{\sigma_i\}} \exp\left\{ -\frac{1}{T}\left(-\sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i \right) \right\} = e^{-F/T}$$

ferromagnetic

$$t = \frac{T - T_c}{T_c}$$



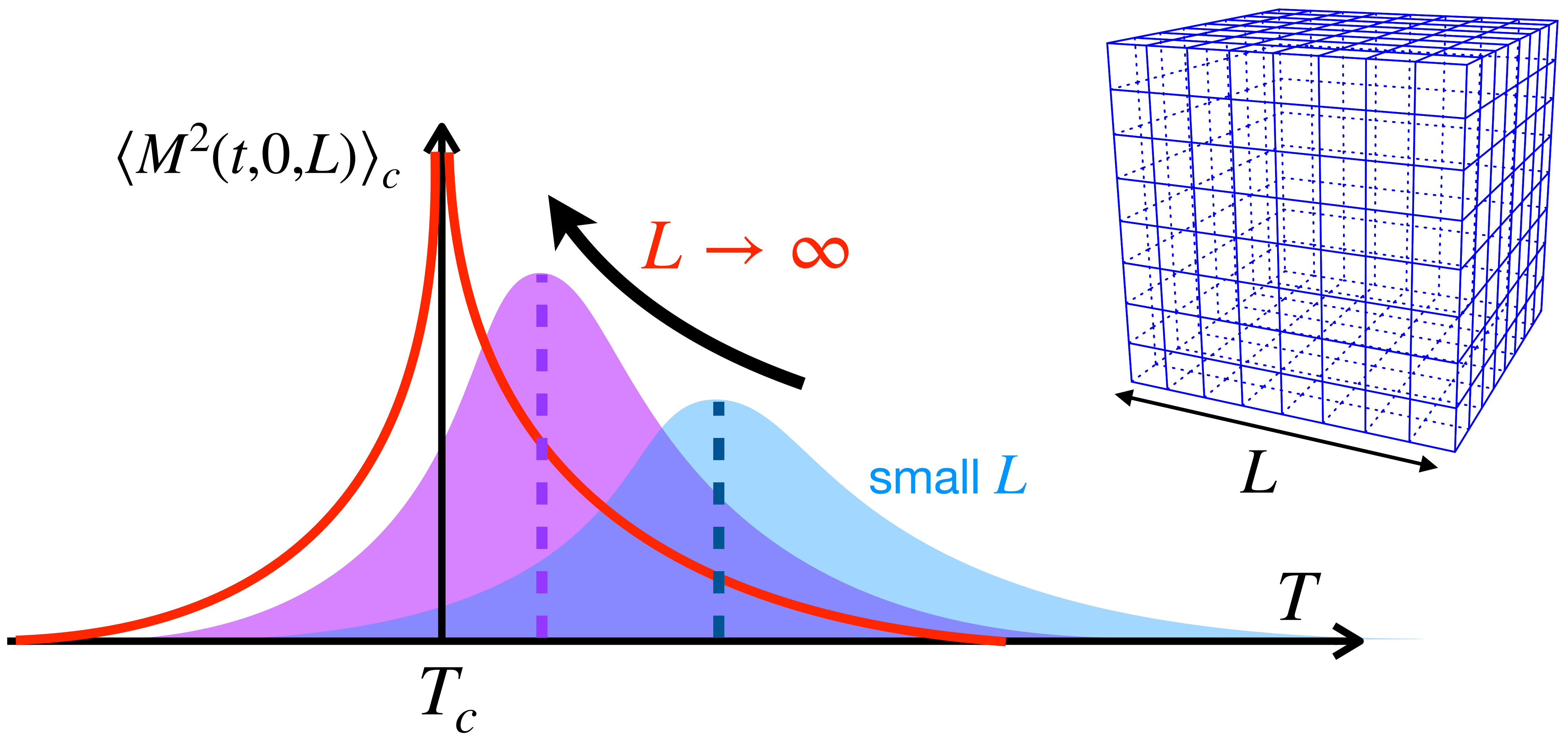
CP at $T = T_c = 4.51152321(4)$

[Ferrenberg+, Phys.Rev. E97 (2018)]
Binder-cumulant method
lattices up to 1024^3

$\langle M \rangle < 0$ $\langle M \rangle > 0$

$$M \equiv \sum_i \sigma_i$$

1st order PT

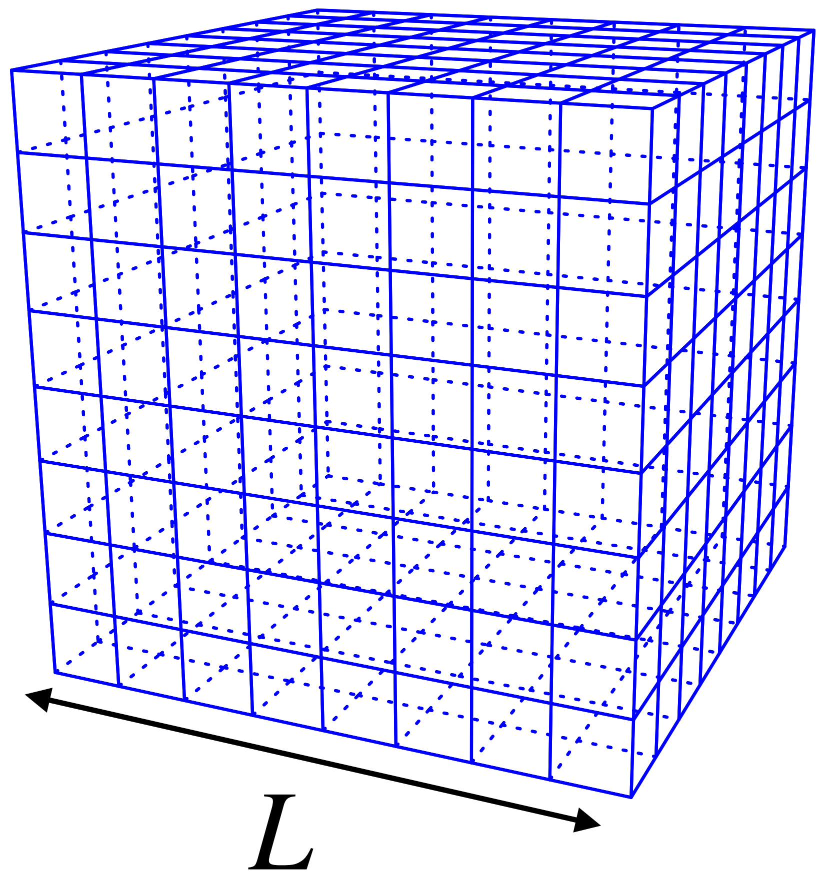


Finite-size scaling

scaling function

Near the CP & at large L, $F(t,h,L) \;\approx\; F_{\text{sing}}(t,h,L) = \tilde{F}(L^{y_t}t, L^{y_h}h).$

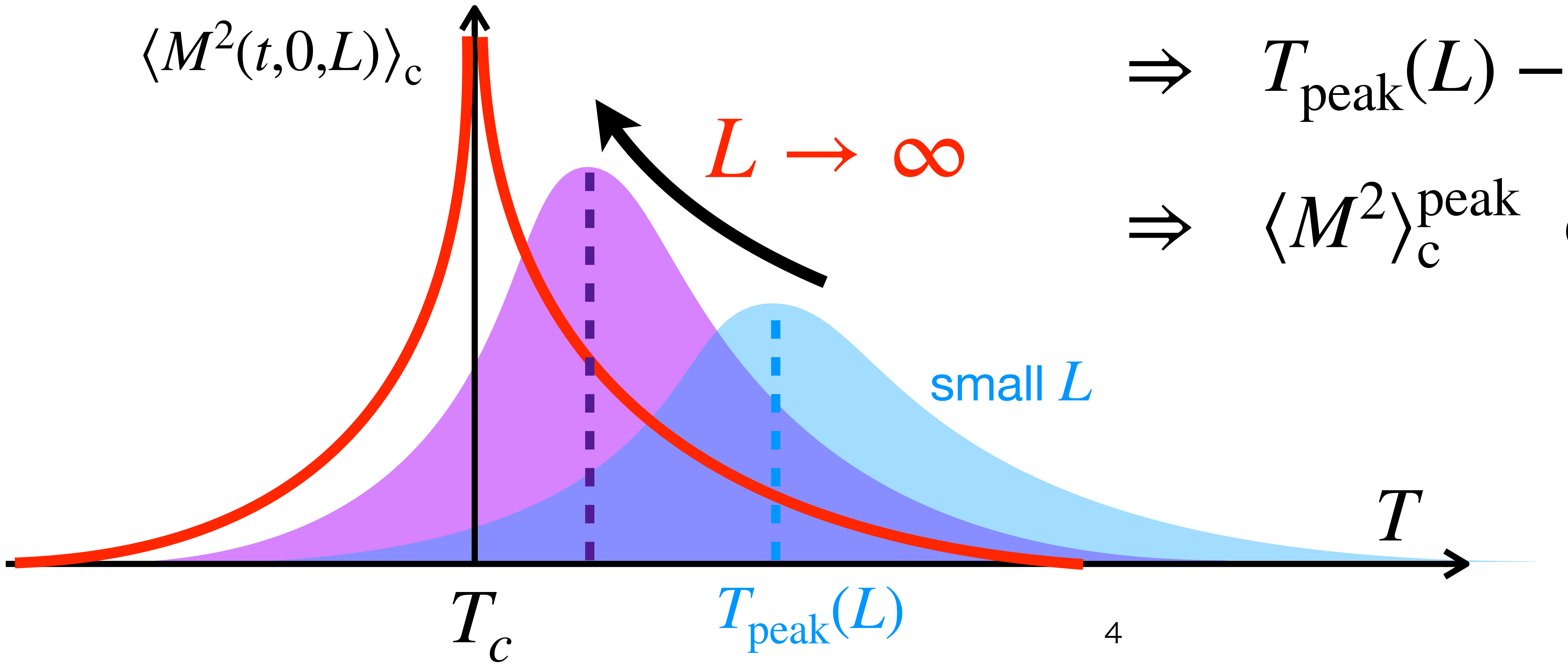
$y_t = 1.58737472(29),\ y_h = 2.481851192(24)$ for 3d Ising



cumulants

$$\langle M^n(t,h,L)\rangle_{\mathrm{c}} \equiv -T^{n-1}\frac{\partial^n F(t,h,L)}{\partial h^n} \approx -L^{-ny_h}\tilde{F}^{(n)}(L^{y_t}t, L^{y_h}h), \quad \tilde{F}^{(n)} \equiv T^{n-1}\partial^n\tilde{F}/\partial h^n$$

$$\Rightarrow \quad \langle M^n(t,0,L)\rangle_{\mathrm{c}} \approx -L^{-ny_h}\tilde{F}^{(n)}(L^{y_t}t,0)$$



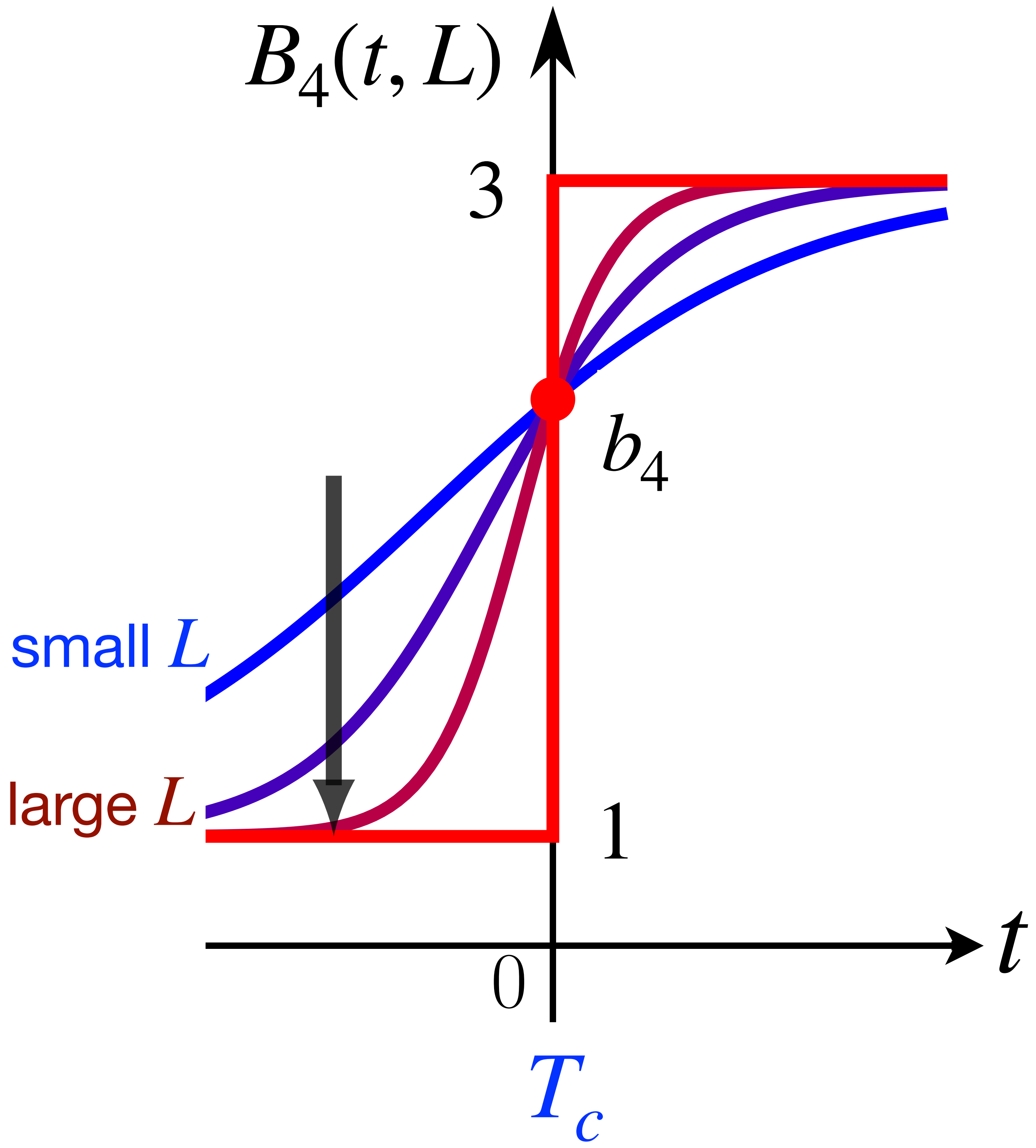
$$\Rightarrow \quad T_{\text{peak}}(L) - T_c \;\propto\; L^{-y_t}$$

$$\Rightarrow \quad \langle M^2\rangle_{\mathrm{c}}^{\text{peak}} \;\propto\; L^{2y_h}$$

Binder cumulant

3d-Ising

$$B_4(t,L) \equiv \frac{\langle M^4(t,0,L)\rangle_c}{\langle M^2(t,0,L)\rangle_c^2} + 3 = b_4 + c_4 L^{y_t} t + O(t^2) \longrightarrow_{L\to\infty} \begin{cases} 3 & (t>0) \\ 1 & (t<0) \end{cases}$$



$\Rightarrow$ T_c at $L \to \infty$ can be determined directly on finite lattices with $L < \infty$.

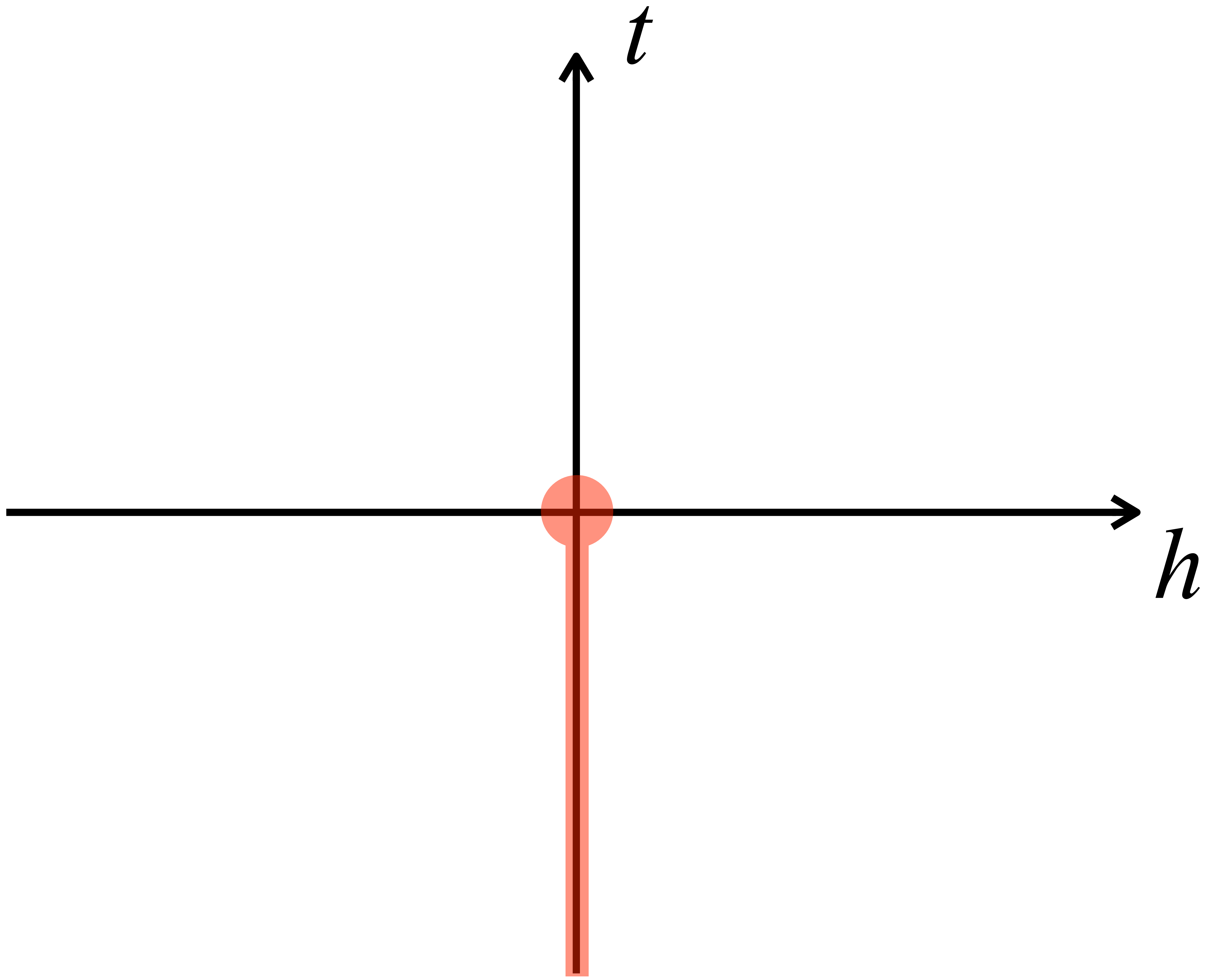
Ferrenberg+, Phys.Rev.E97 (2018), lattices up to 1024^3

$$T_c = 4.51152321(4)$$

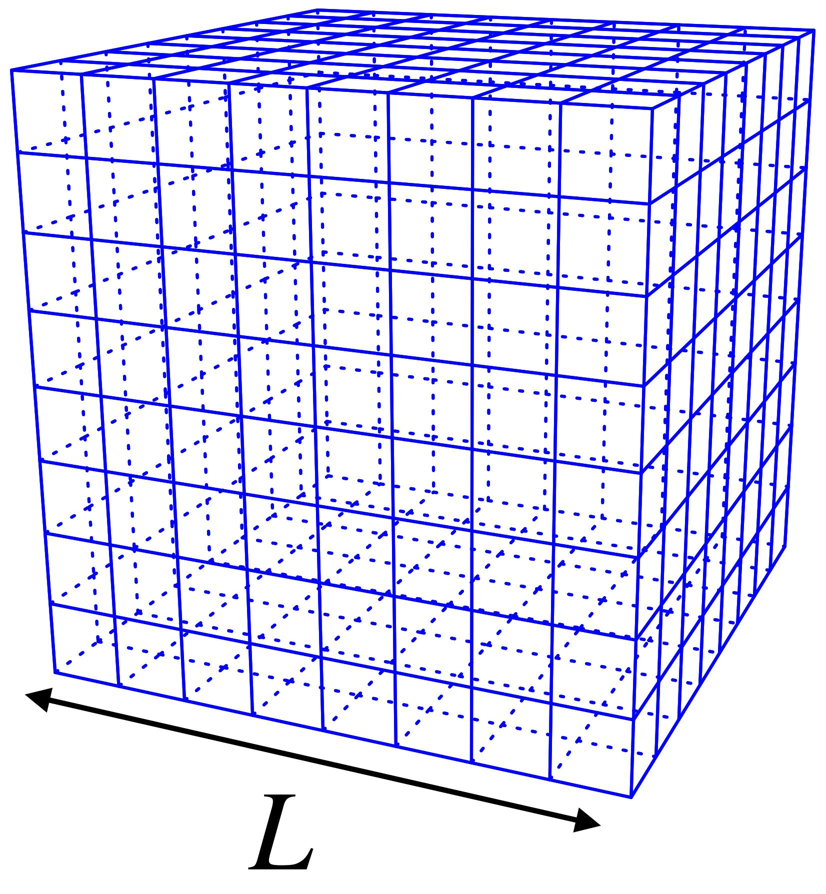
$$b_4 = 1.60356(15)$$

Lee-Yang Zero

3d-Ising



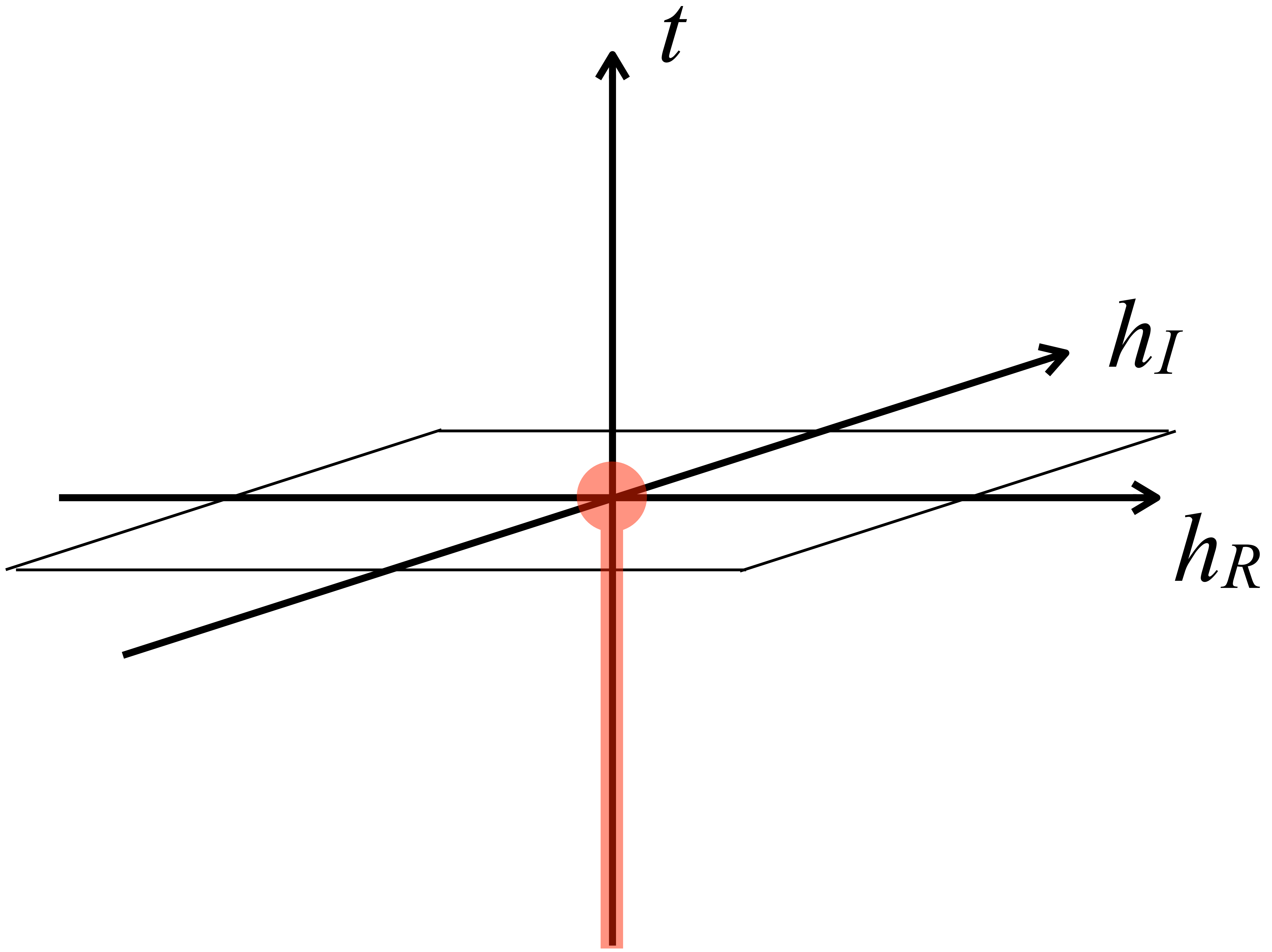
$$Z = \sum_{\{\sigma_i\}} e^{-\frac{1}{T}\left(-\sum_{\langle i,j\rangle}\sigma_i\sigma_j - h\sum_i \sigma_i\right)}$$



On ***finite lattices***, $Z > 0$ and regular for real t, h

Lee-Yang Zero

3d-Ising



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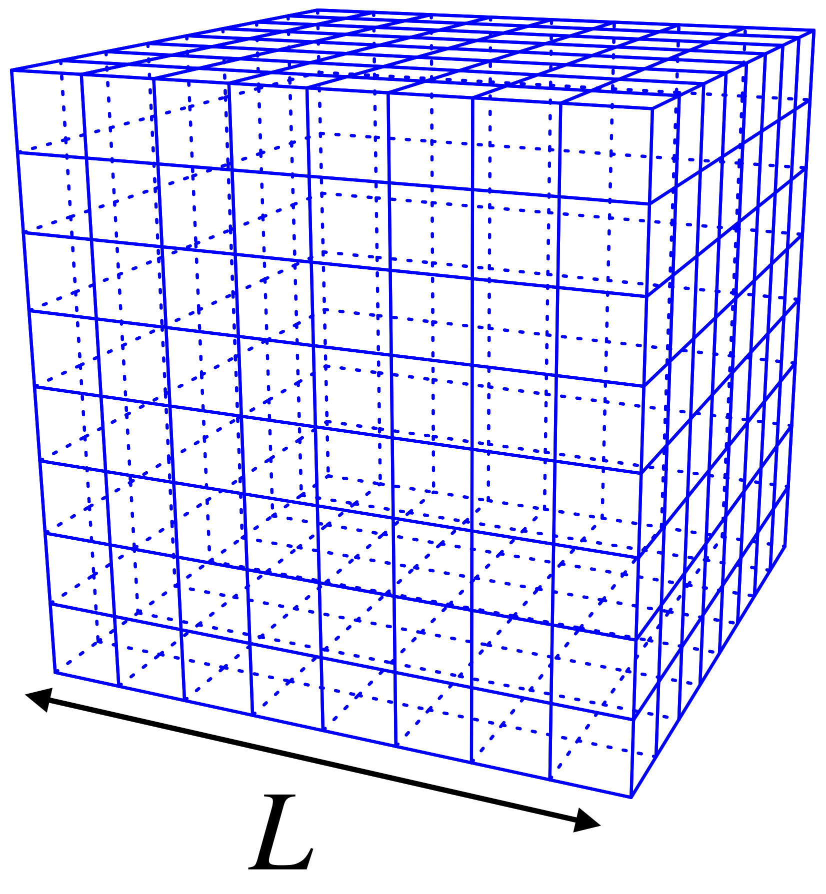


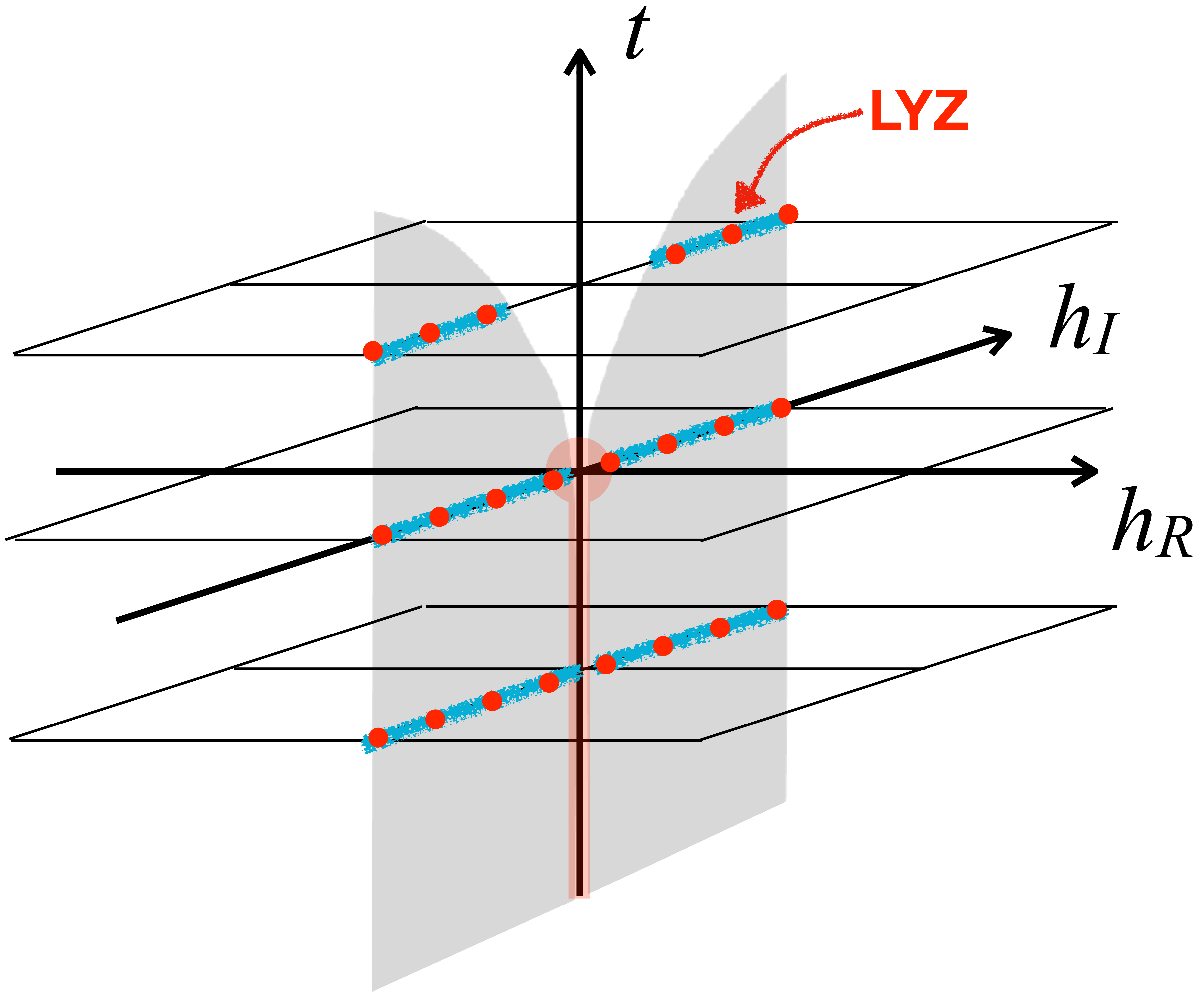
On *finite lattices*, $Z > 0$ and regular for real t, h but can have $Z = 0$ for *complex* t, h.

LYZ: zero of $Z(t, h, L)$ for complex h

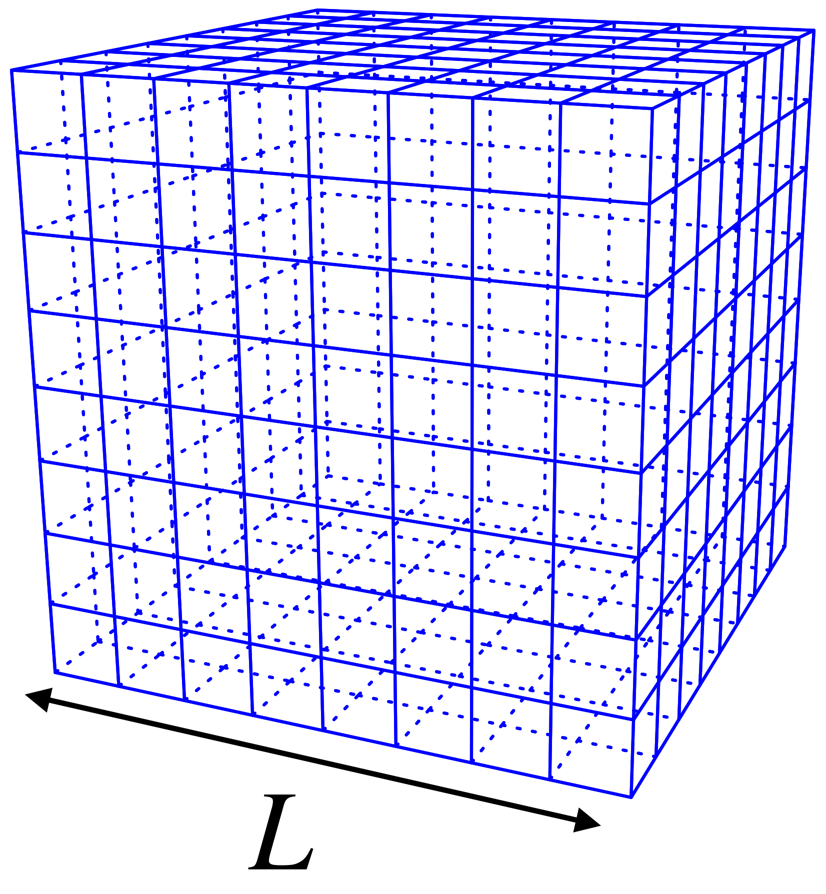
$$h \longrightarrow h_R + ih_I$$

Lee-Yang Zero

3d-Ising



$$Z = \sum_{\{\sigma_i\}} e^{-\frac{1}{T}\left(-\sum_{\langle i,j\rangle}\sigma_i\sigma_j - h\sum_i \sigma_i\right)}$$



On ***finite lattices***, $Z > 0$ and regular for real t, h but can have $Z = 0$ for complex t, h.

LYZ: zero of $Z(t, h, L)$ for complex h

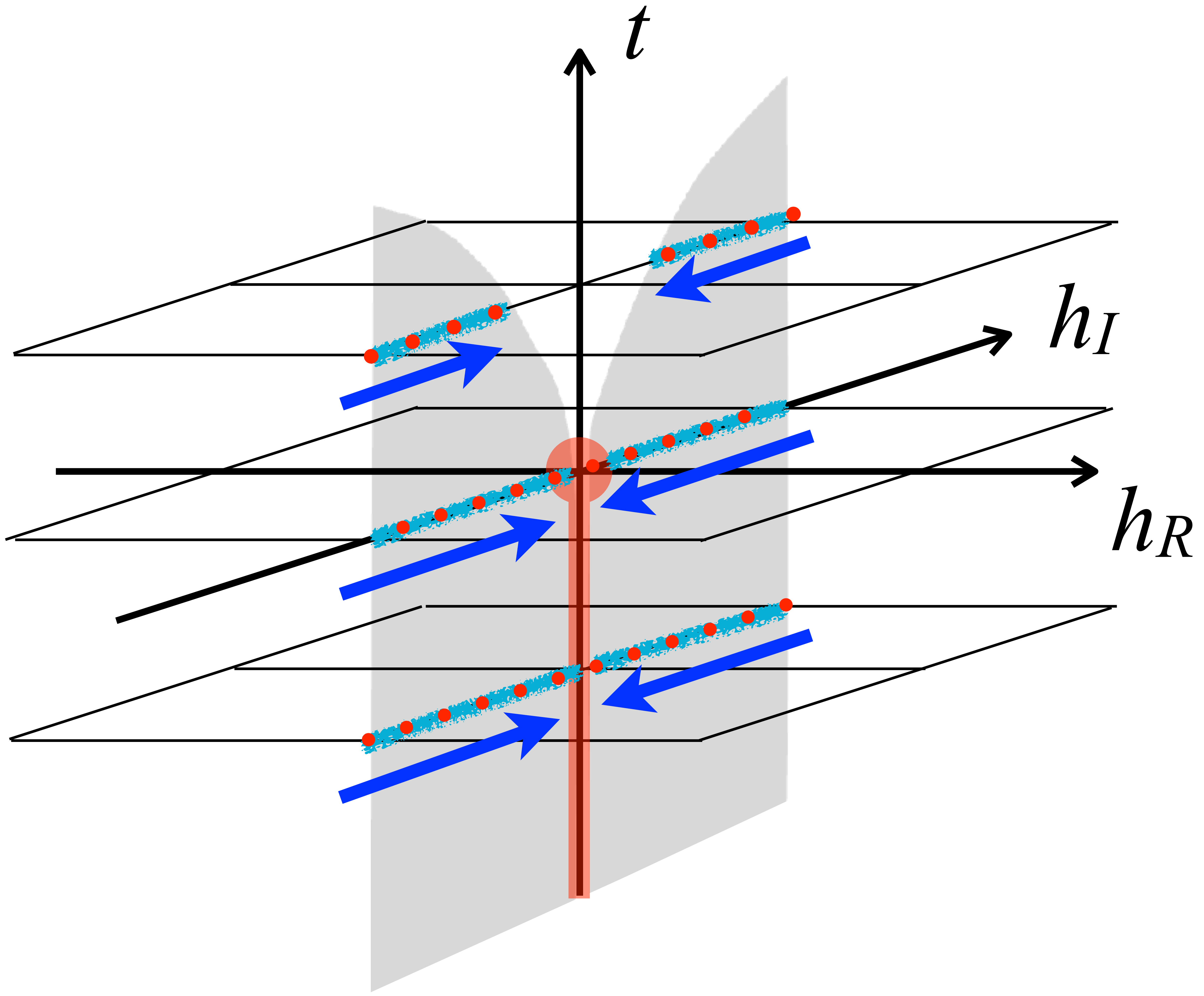
$$h \longrightarrow h_R + ih_I$$

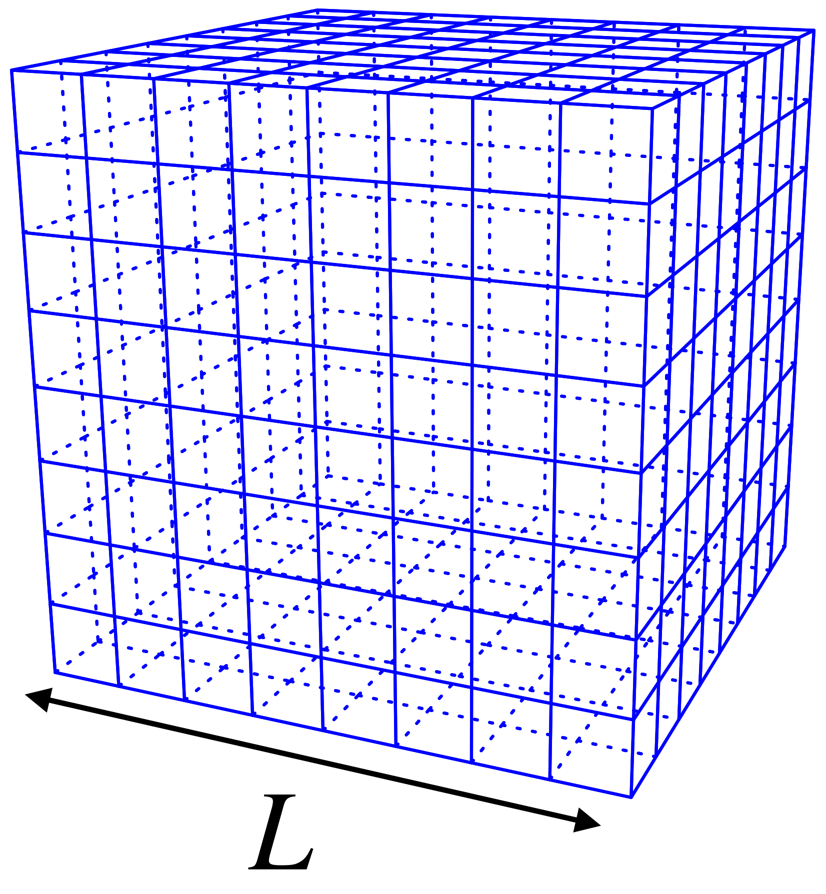
Known to distribute on the imaginary h axis.

[Yang-Lee (1952)]

Lee-Yang Zero

3d-Ising





As $L \to \infty$, the distribution becomes denser.

- ☑ $t < 0 \Rightarrow$ **pinches the real axis**
  indicating the 1st order transition at $L = \infty$
- ☑ $t > 0 \Rightarrow$ make a couple of lines
  leaving a **gap around the real axis**

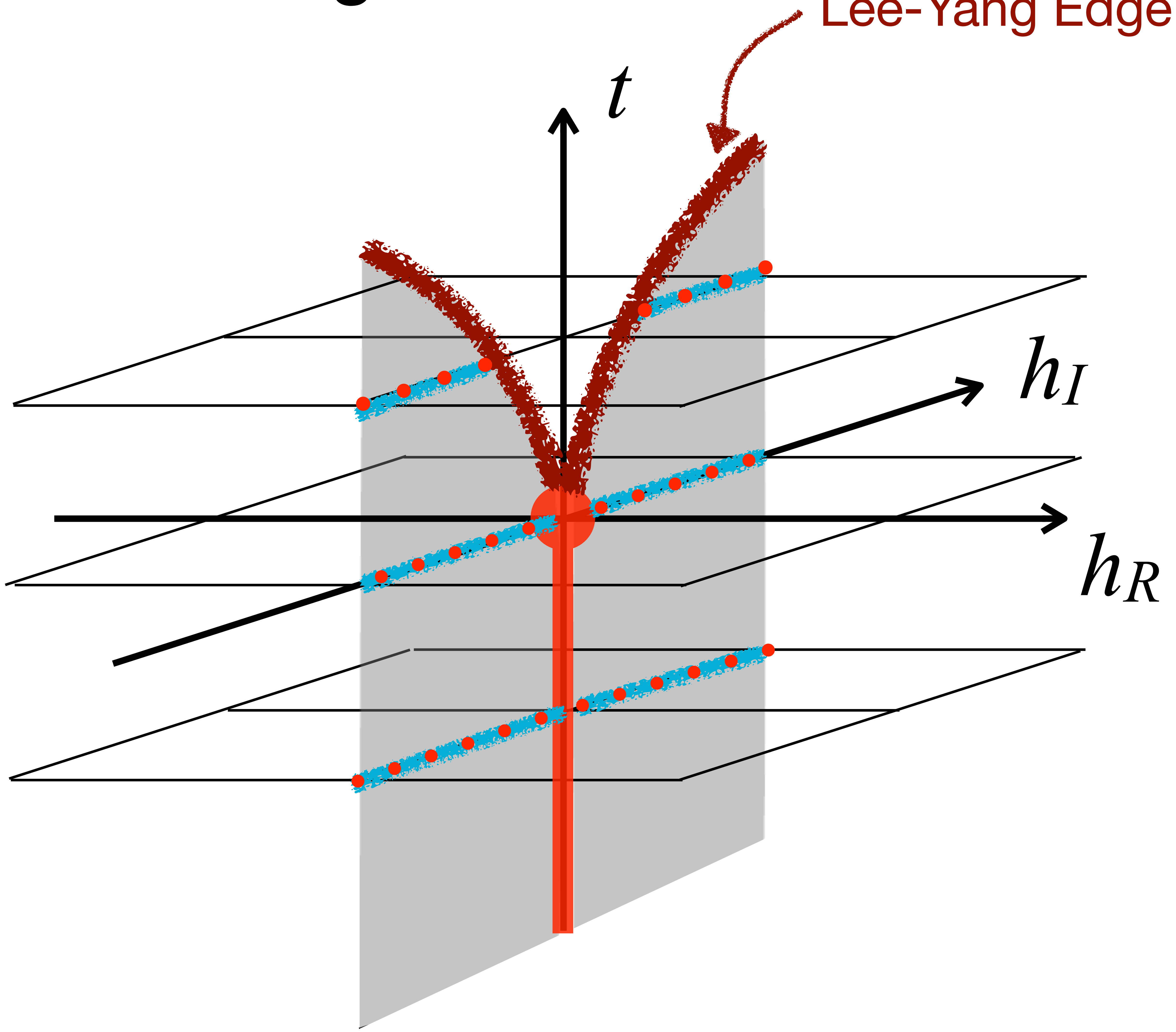
n^{th} **LYZ** from the real axis:

$$h_{\mathrm{LY}}^{(n)}(t, L) \to_{L\to\infty} \begin{cases} h_{\mathrm{LYES}}(t) \neq 0 & (t > 0) \\ a(t)\,\dfrac{2n-1}{L^3} \longrightarrow 0 & (t < 0) \end{cases}$$

Lee-Yang Edge Singularity

3d-Ising

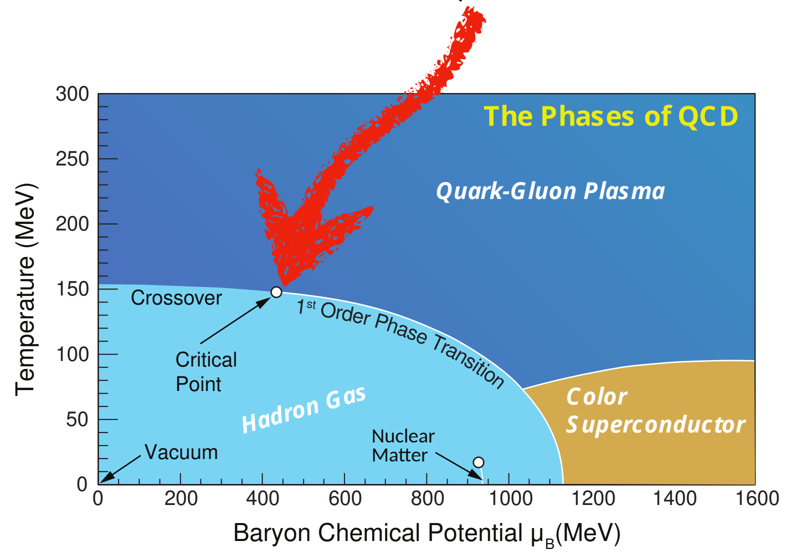
Lee-Yang Edge Singularity



$h_{\text{LYES}} \propto t^{y_h/y_t}$ from the scaling relation in the $L \to \infty$ limit

[Stephanov (2006)]

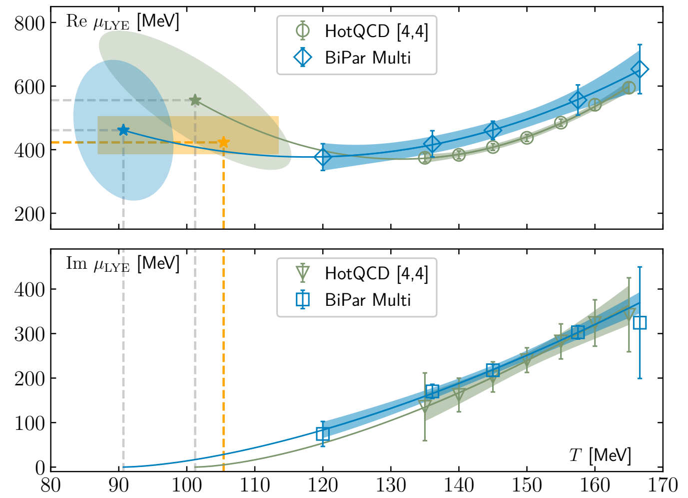
Application to QCD assuming $h_{\text{LY}}^{(1)} \approx h_{\text{LYES}}$ at finite L



[Basar, PRC 110, 074511 (2024)]

HotQCD Taylor coeff's, $N_{\text{F}} = 2 + 1$ HISQ, $N_t = 8 - 12$

$T^{\text{CEP}} \approx 100\,\text{MeV}, \;\; \mu_{\text{B}}^{\text{CEP}} \approx 580\,\text{MeV}$



[Clarke+, arXiv:2405.10196 (2024)]

$N_{\text{F}} = 2 + 1$ HISQ, $N_t = 6$, $N_s = 36$, imag. μ_{B}

$T^{\text{CEP}} = 105^{+8}_{-18}\text{MeV}, \;\; \mu_{\text{B}}^{\text{CEP}} = 422^{+80}_{-35}\text{MeV}$

[Adam+, arXiv:2502.03211 (2025)]

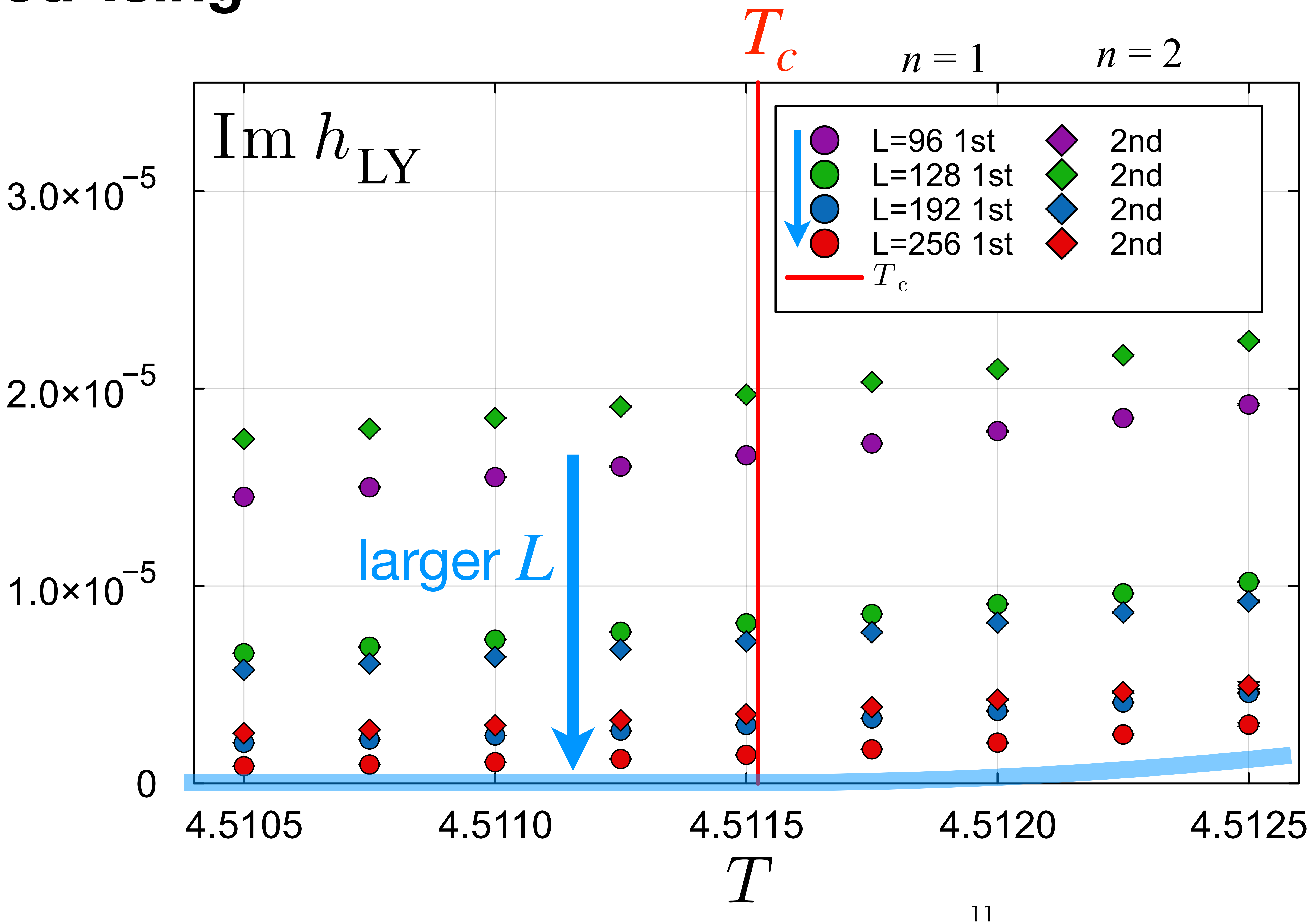
Wuppertal-Budapest Taylor coeff's, $N_{\text{F}} = 2 + 1$, $N_t = 8$, $N_s = 16$

$T^{\text{CEP}} < 103\,\text{MeV}, \;\; \text{or no CEP}$

Lee-Yang Zero

3d-Ising

[Wada-Kitazawa-KK, arXiv:2508.20422 → JPSJ]

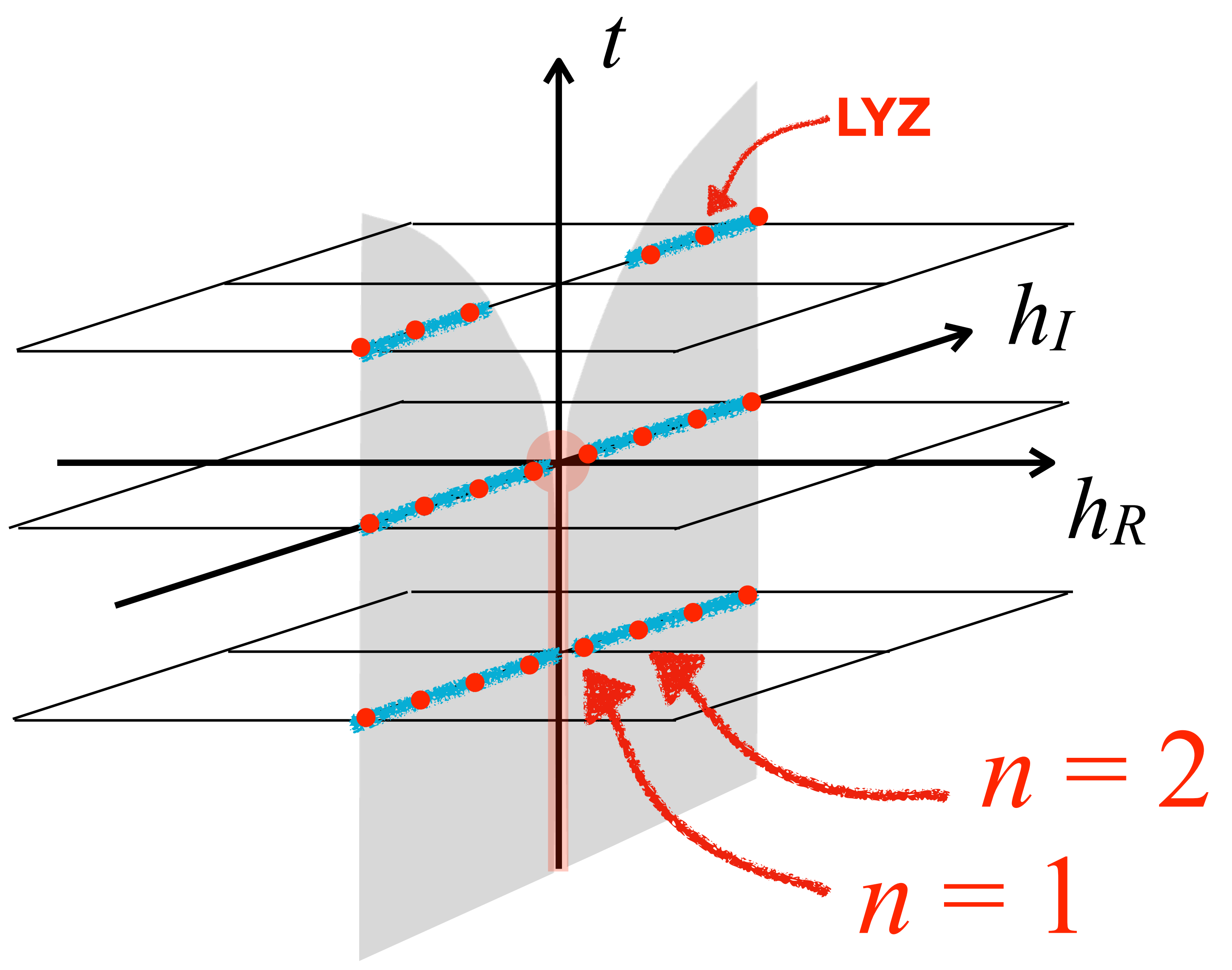


Wolf cluster algorithm + reweighting

$L = 16, \cdots 192, 256$

3 T-values around T_C

each 2×10^6 meas. every 10 MC steps



To get T_c

⇒ need to study the L-dep.

Finite-size scaling of LYZs

scaling functions

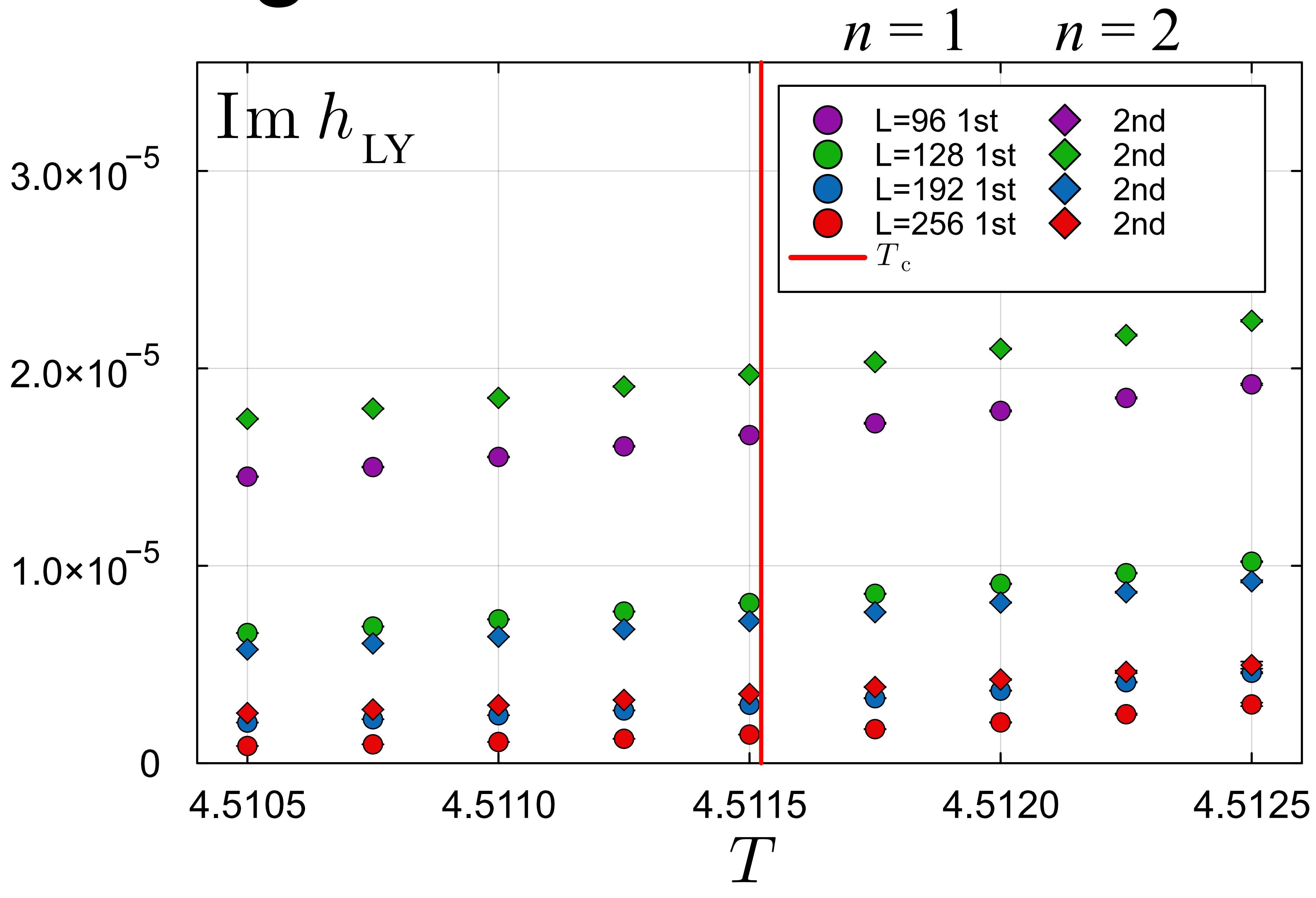
Near the CP, $F(t,h,L) \approx F_{\mathrm{sing}}(t,h,L) = \tilde{F}(L^{y_t}t, L^{y_h}h) \quad \Rightarrow \quad Z \sim Z_{\mathrm{sing}}(t,h,L) = \tilde{Z}(L^{y_t}t, L^{y_h}h)$

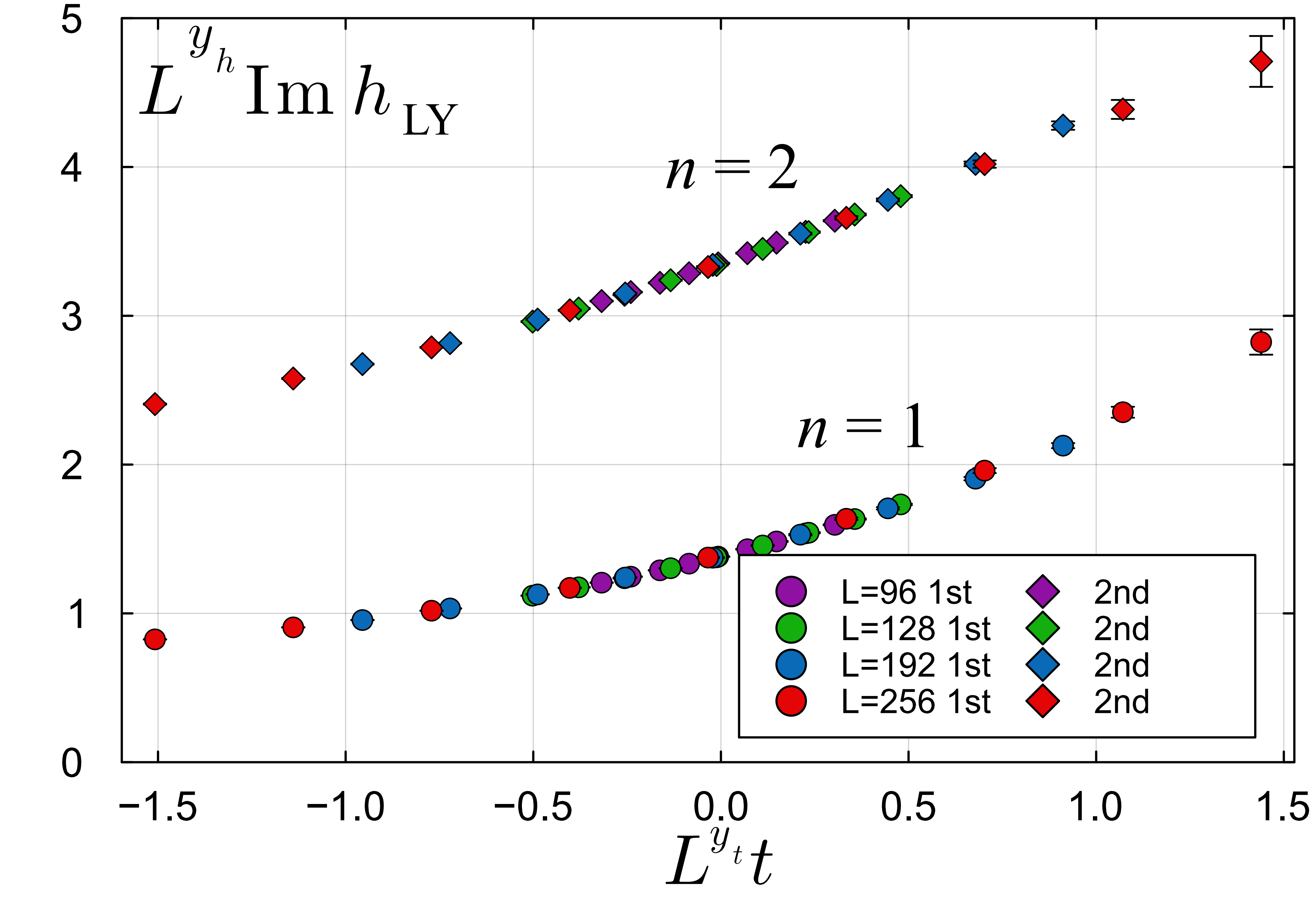
LYZ: $\Rightarrow \boxed{L^{y_h} h_{\mathrm{LY}}^{(n)}(t,L) = \tilde{h}_{\mathrm{LY}}^{(n)}(L^{y_t}t)}$

[Itzykson (1983)]

3d-Ising LYZs

[Wada-Kitazawa-KK, arXiv:2508.20422 → JPSJ]





Lee-Yang Zero Ratio

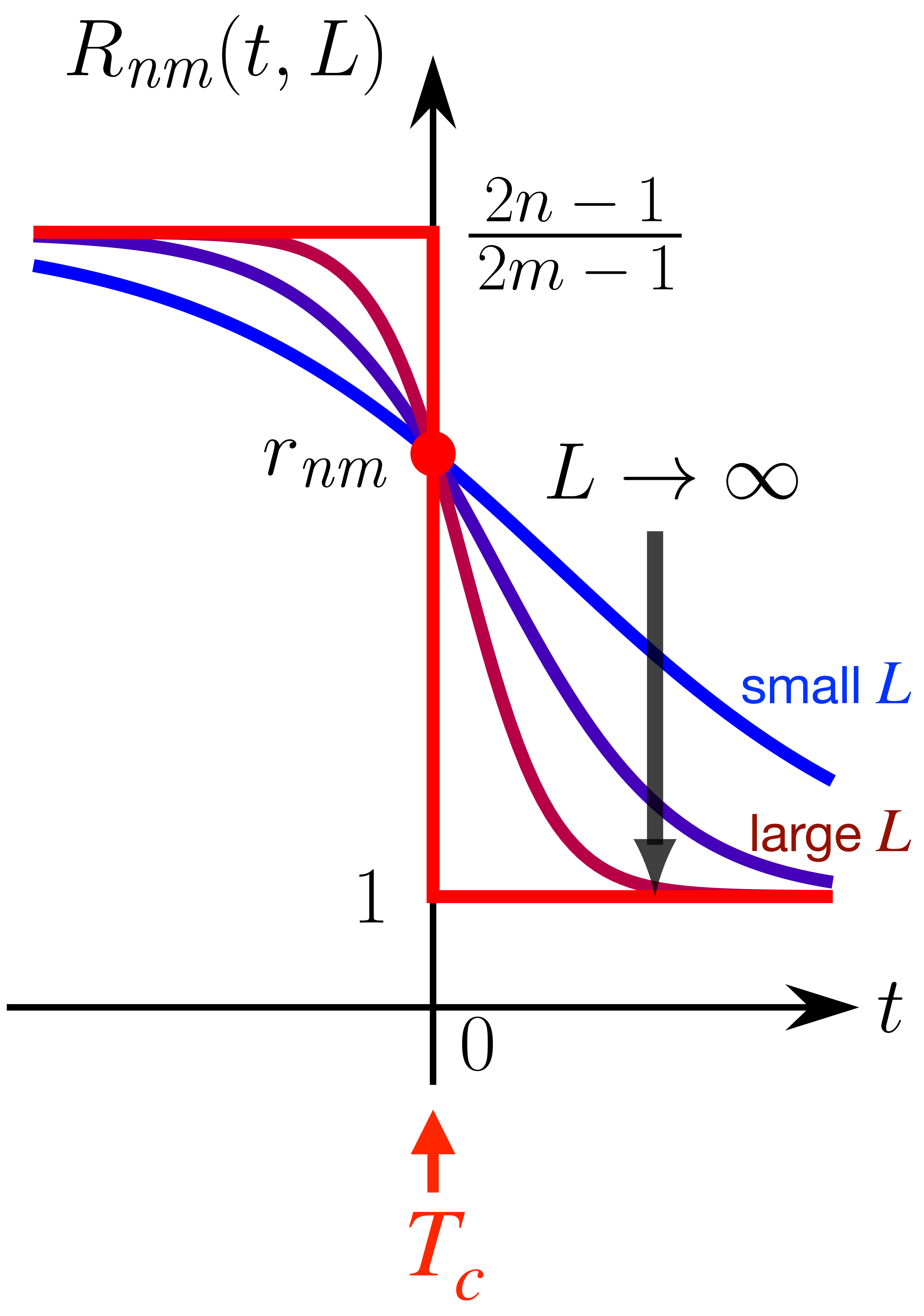
[Wada-Kitazawa-KK, PRL 134, 162302 (2025)]

$$R_{nm}(t,L) \equiv \frac{\mathrm{Im}\, h_{\mathrm{LY}}^{(n)}(t,L)}{\mathrm{Im}\, h_{\mathrm{LY}}^{(m)}(t,L)} = r_{nm} + c_{nm} L^{y_t} t + O(t^2)$$

$$L^{y_h} h_{\mathrm{LY}}^{(n)}(t,L) = \tilde{h}_{\mathrm{LY}}^{(n)}(L^{y_t} t) = X_n + Y_n L^{y_t} t + O(t^2)$$

$$\xrightarrow[L\to\infty]{} \begin{cases} 1 & (t>0) \\ \dfrac{2n-1}{2m-1} & (t<0) \end{cases}$$

$$h_{\mathrm{LY}}^{(n)}(t,L) \xrightarrow[L\to\infty]{} \begin{cases} h_{\mathrm{LYES}}(t) & (t>0) \\ a(t)\dfrac{2n-1}{L^3} \longrightarrow 0 & (t<0) \end{cases}$$



Binder cumulant

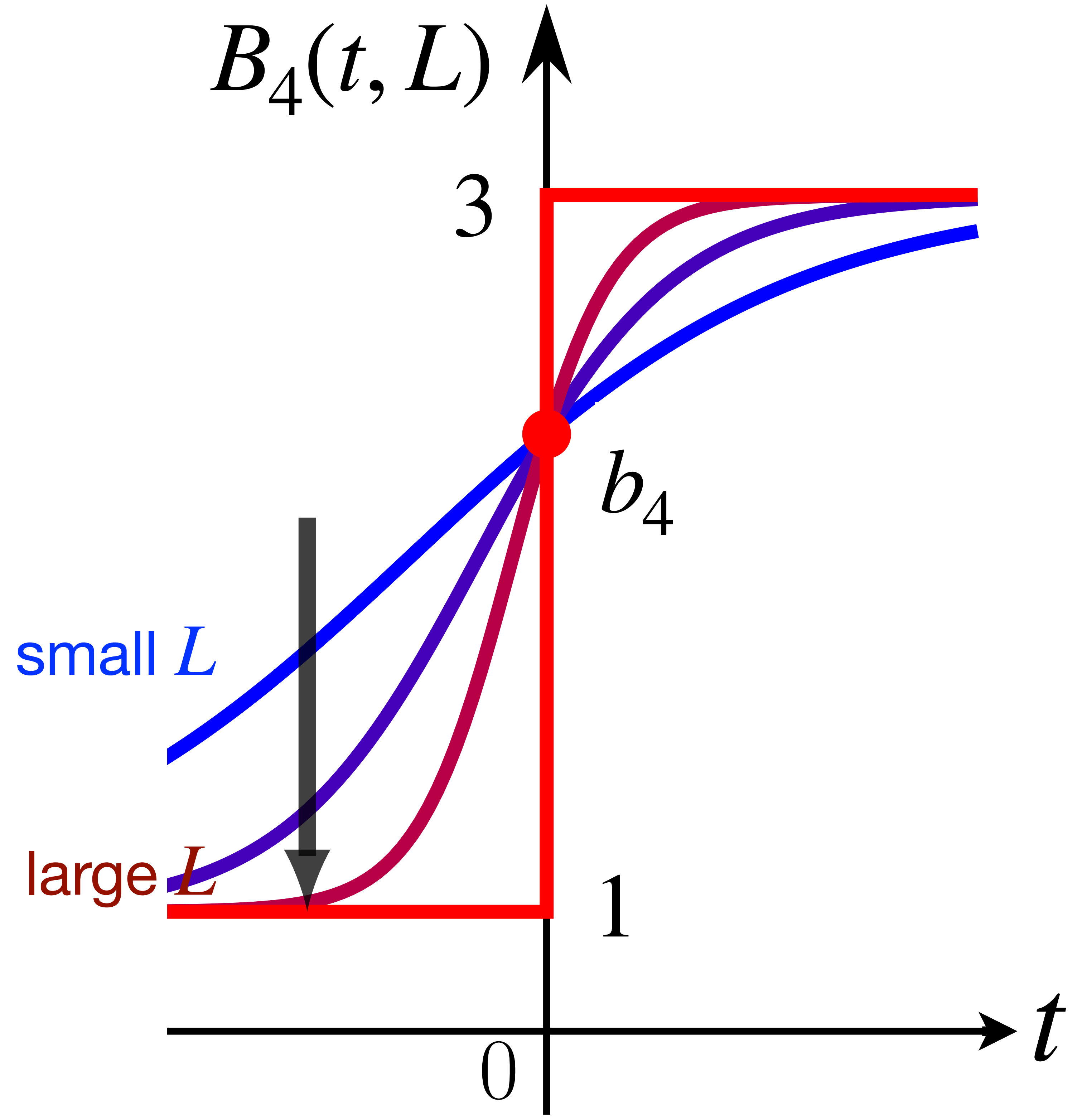
$$B_4(t,L) \equiv \frac{\langle M^4(t,0,L)\rangle_c}{\langle M^2(t,0,L)\rangle_c^2} + 3 = b_4 + c_4 L^{y_t} t + O(t^2) \longrightarrow_{L\to\infty} \begin{cases} 3 & (t>0) \\ 1 & (t<0) \end{cases}$$

- **LYZR** provides us with

  a new convenient & independent tool

  to determine CP

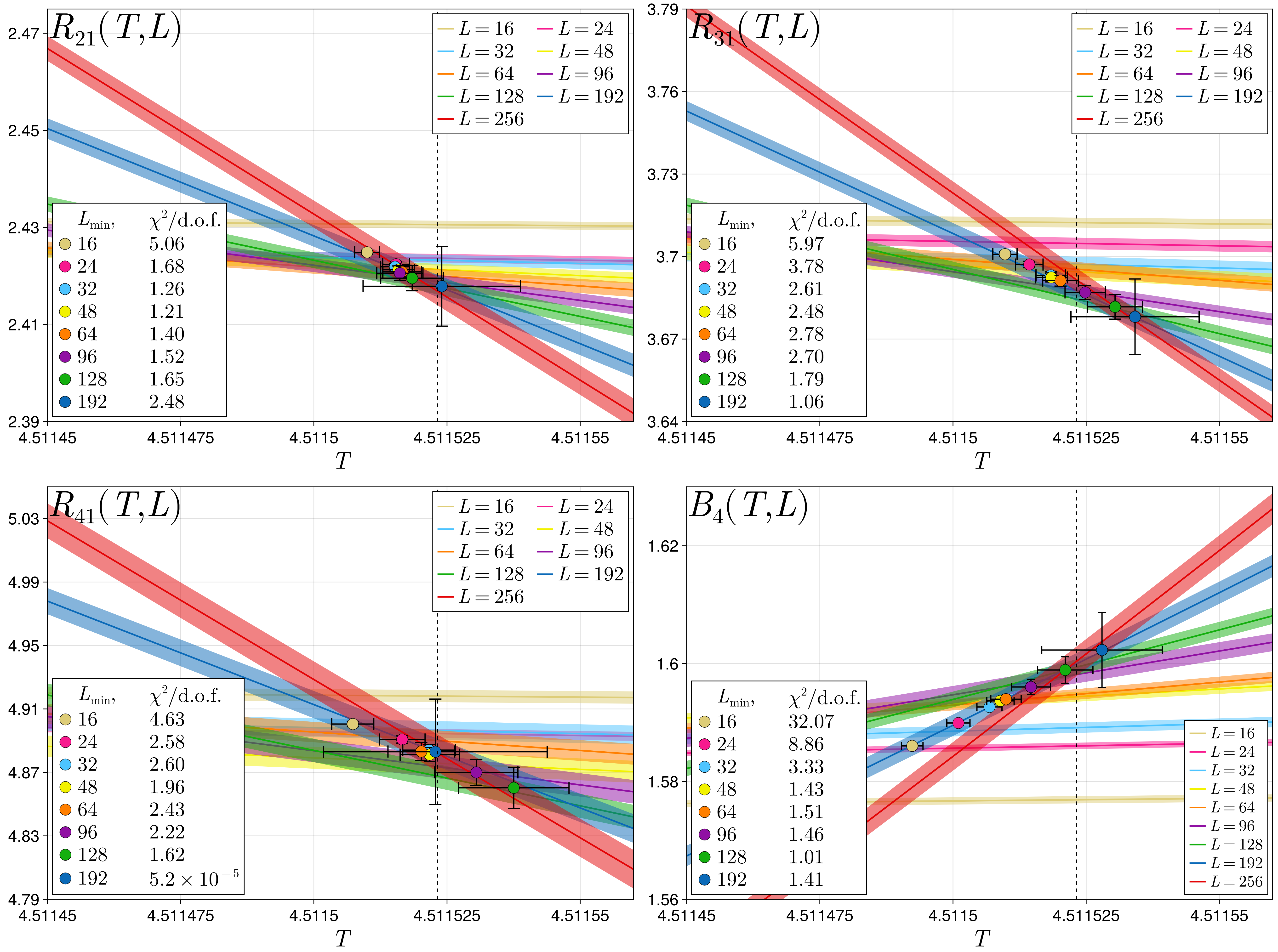
  by intersection analyses *á la* Binder-cumulant.



Lee-Yang Zero Ratio

[Wada-Kitazawa-KK, arXiv:2508.20422 → JPSJ]

3d-Ising



		Tc	y_t
Our results	R_{21}	4.5115185(59)	1.5874(115)
	B_4	4.5115211(52)	1.5691(86)
Ferrenberg+ (2018)	B_4	4.5115232(1)	1.58723(22)

- LYZR consistent with known T_c and y_t
- LYZR is as powerful as B_4
- LYZR works with smaller L's than B_4

Sub-leading contribution in FSS

3d-Ising

[Wada-Kitazawa-KK, arXiv:2508.20422 → JPSJ]

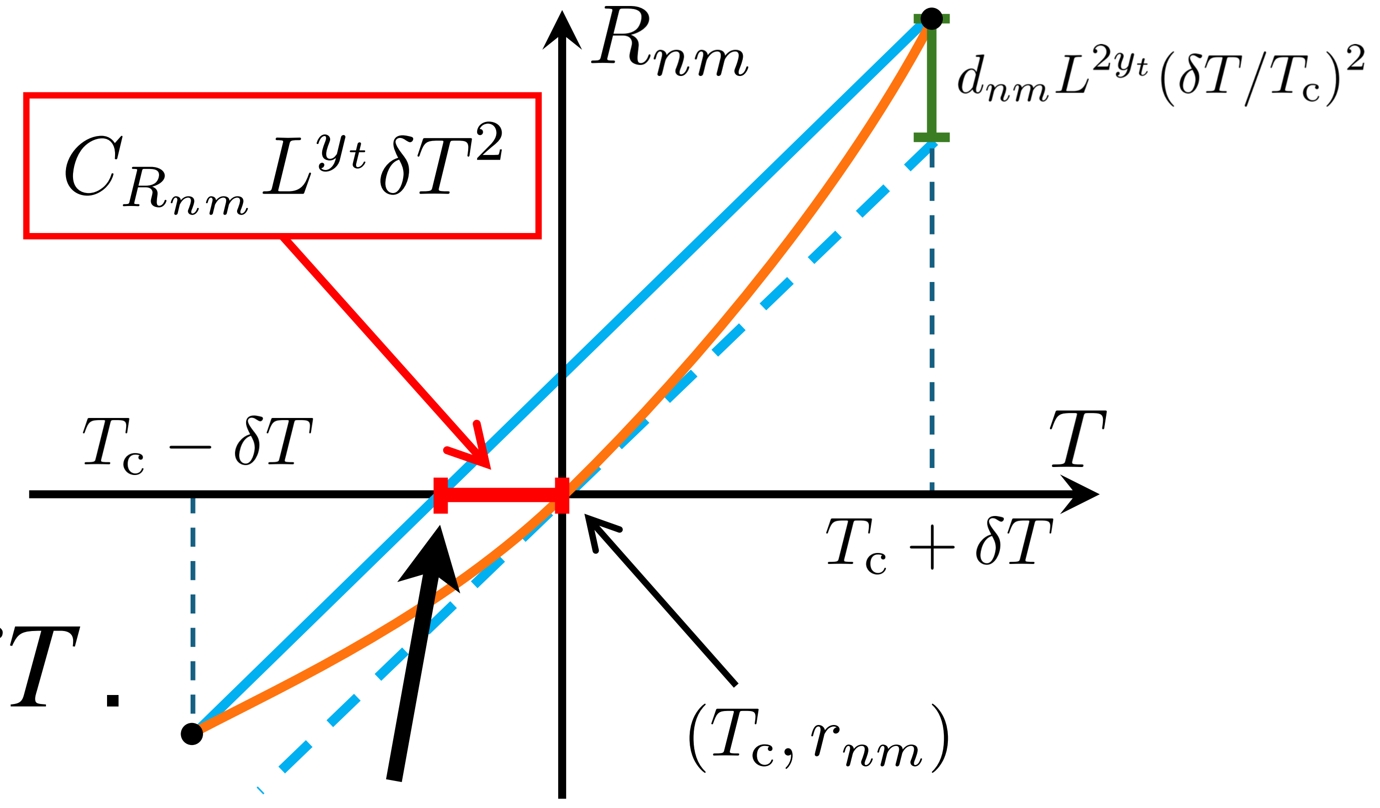
FSS fits: $R_{n1}(T,L) = r_{n1} + c_{n1}L^{y_t}t + d_{n1}L^{2y_t}t^2,\ \ t = \frac{T-T_c}{T_c}$ with 5 fitting param's [similar for B_4 too.]

The next-to-linear term not negligible in our fits,
while large non-linearity not desirable.

Normalized curvature as a measure of the non-linearity:

$$C_f \equiv L^{-y_t} \frac{\partial^2 f/\partial T^2}{\partial f/\partial T}\bigg|_{T=T_C}, \quad f = R_{nm}(T,L) \text{ or } B_4(T,L)$$

$\Rightarrow$ $C_f L^{y_t} \delta T^2$ = **error in** T_c obtained by linear approx. of f's at $T = T_c \pm \delta T$.



T_c by linear approx. of R_{nm}

f	R_{21}	R_{31}	R_{41}	B_4
C_f	0.0093(17)	0.0053(17)	-0.0010(34)	0.0364(37)

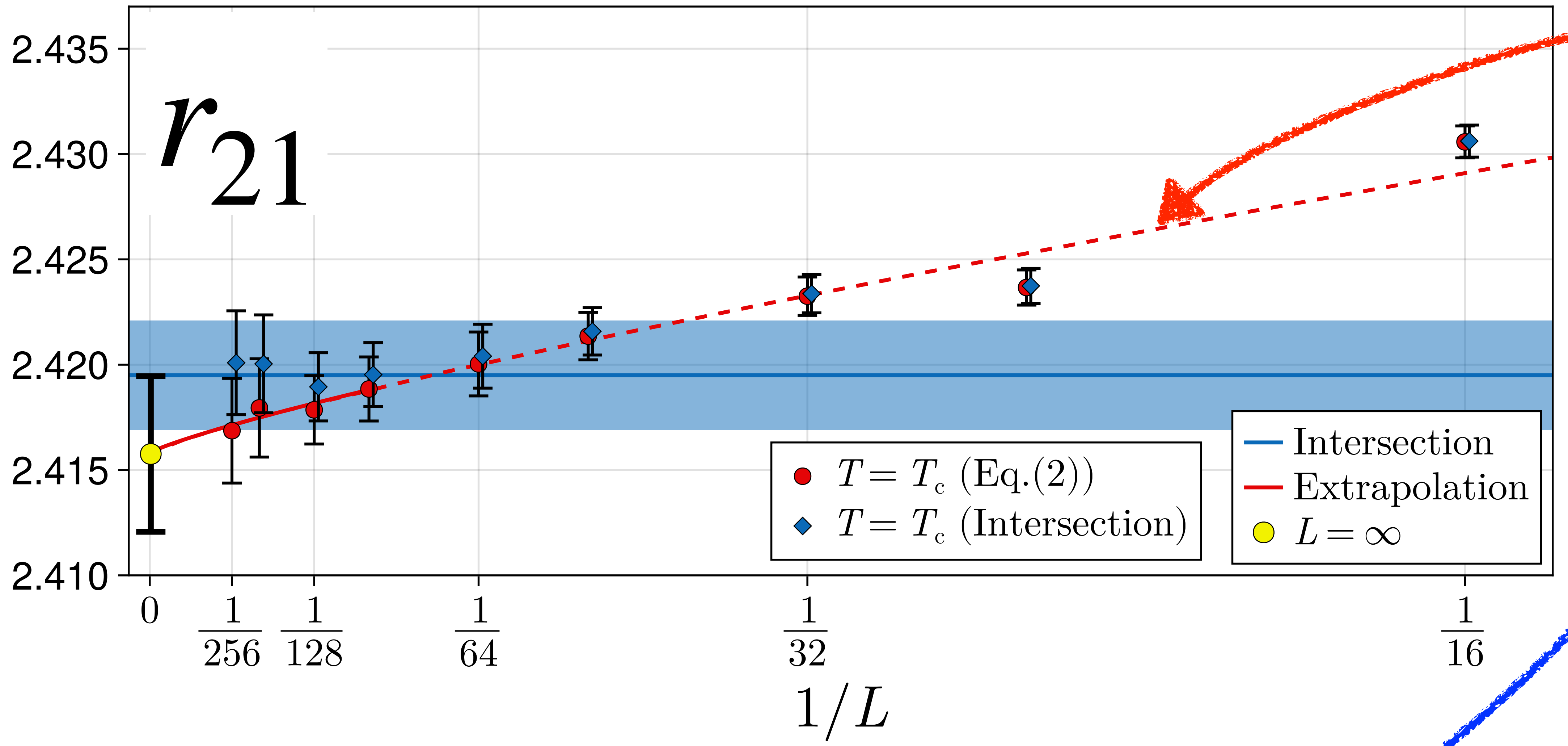
LYZR has smaller sub-leading than B_4.

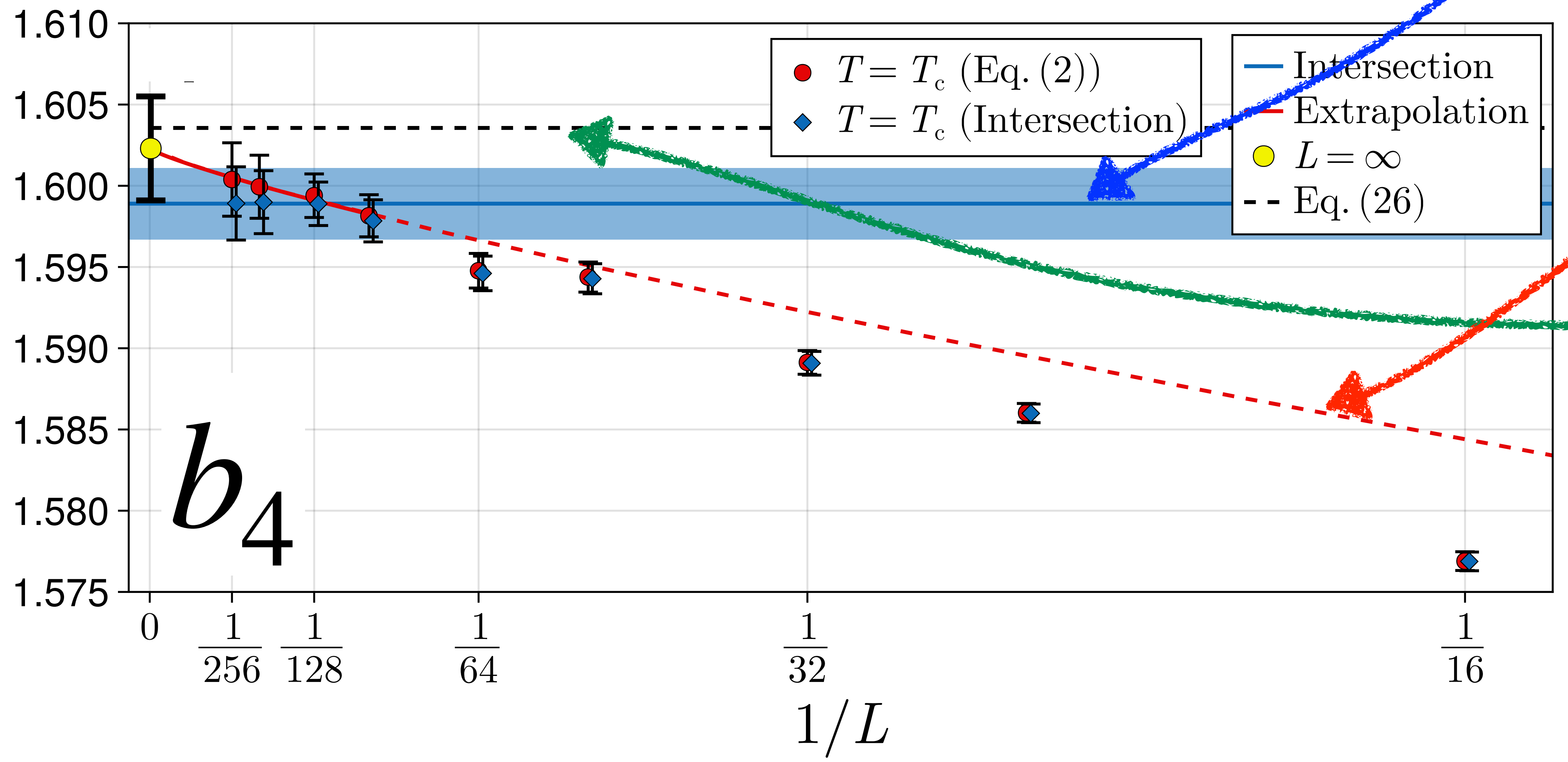
Corrections to FSS

[Wada-Kitazawa-KK, arXiv:2508.20422 → JPSJ]

3d-Ising

R_{nm} and B_4 at $T_c = 4.51152321(4)$ [Ferrenberg+ (2018)]





Fit with FSS-correction due to dominant irrelevant op.

$$R_{21}(T_c, L) = r_{21}\left(1 + cL^{-\omega_1}\right)$$

$\omega_1 = 0.8303(18)$ [El-Showk+. (2014)]

$\Rightarrow r_{21} = 2.4158(37)(5)$

$\updownarrow$ consistent !

FSS fit using $L = 128$-256 $\Rightarrow$ $\left[\begin{array}{l} r_{21} = 2.4195(26) \\ b_4 = 1.59892(224) \end{array}\right.$

$\updownarrow$ 2σ discrepant !

Fit with FSS-correction $\Rightarrow b_4 = 1.6023(32)(8)$

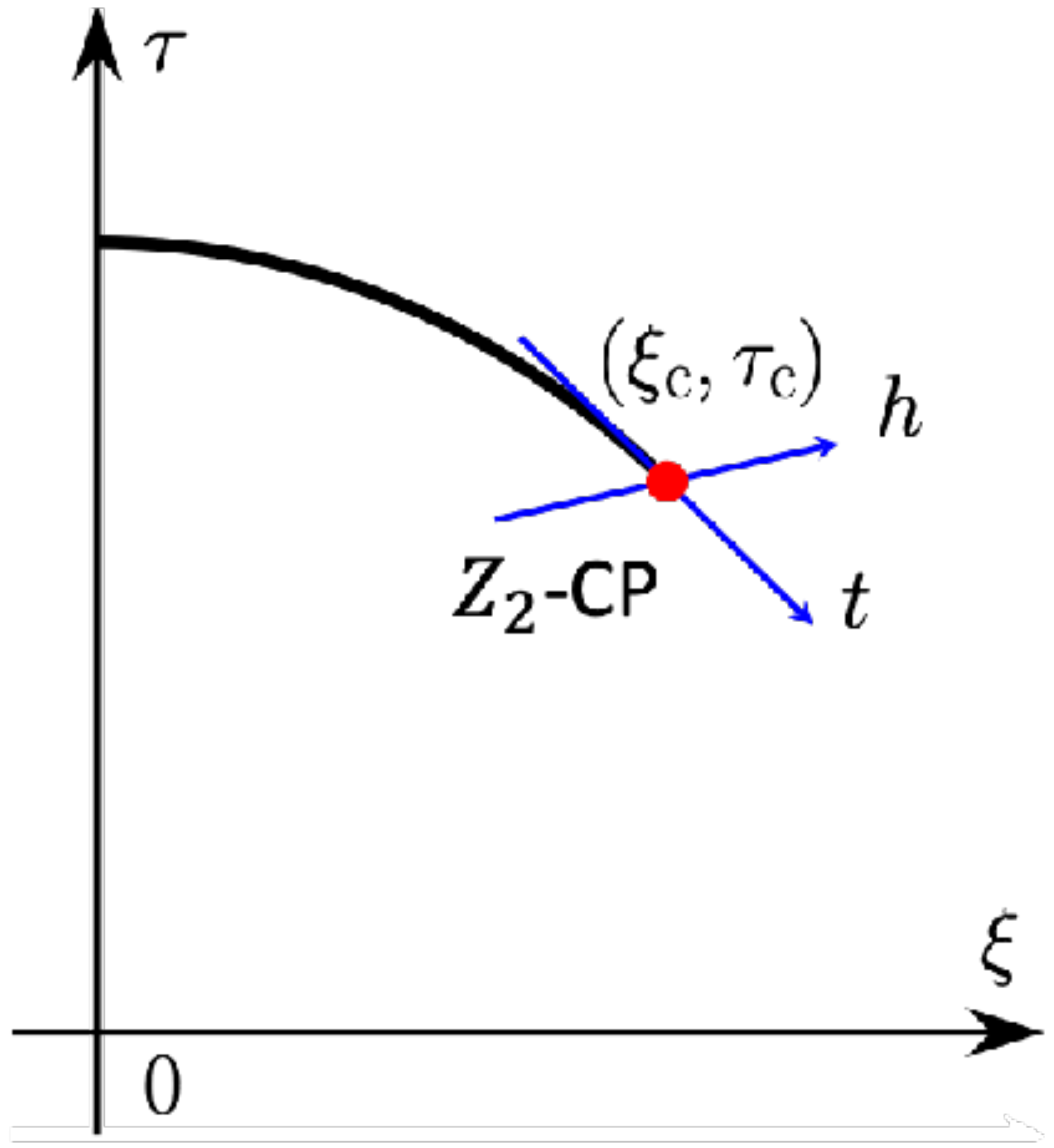
$\updownarrow$ consistent !

[Ferrenberg+ (2018)] $b_4 = 1.60356(15)$

- **LYZR has smaller corrections than B_4.**

[Wada-Kitazawa-KK, PRL 134, 162302 (2025)]

General systems in the same universality class



$H = \tau$ "energy" $+ \xi$ "magnetization"

Map $(t, h) \sim (\tau, \xi)$ by linear approx. around the CP:

$$\begin{pmatrix} t \\ h \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix}$$

$$L^{y_h} h_{\mathrm{LY}}^{(n)}(t, L) = \tilde{h}_{\mathrm{LY}}^{(n)}(L^{y_t} t) = X_n + Y_n L^{y_t} t + O(t^2)$$

LYZ for real τ & complex ξ

$$\begin{cases} \xi_R^{(n)} = \xi_c - \frac{a_{21}}{a_{22}} \delta\tau + \mathcal{O}\left(L^{2\bar{y}}\right) \\ \xi_I^{(n)} = \frac{X_n}{a_{22}} L^{-y_h} + \frac{\det A \; Y_n}{a_{22}^2} \delta\tau L^{\bar{y}} + \mathcal{O}\left(L^{2\bar{y}}\right) \end{cases}$$

$\bar{y} \equiv y_t - y_h \approx -0.89 < 0$

correction due to the mixing

$$\mathscr{R}_{nm}(\tau, L) \equiv \frac{\xi_I^{(n)}(\tau, L)}{\xi_I^{(m)}(\tau, L)} = \left(r_{nm} + C_{nm} L^{y_t} \delta\tau + O(\delta\tau^2)\right) \left(1 + D_{nm} L^{2\bar{y}} + O(L^{4\bar{y}})\right)$$

$C_{nm} = c_{nm} \det A / a_{22}, \;\; D_{nm} = -(Y_n^2 - Y_m^2)\, a_{12}^2 / a_{22}^2$

r_{nm} **is universal**

General systems

[Wada-Kitazawa-KK, PRL 134, 162302 (2025)]

LYZR vs. Binder

Ising FSS + mapping $\begin{pmatrix} t \\ h \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix}$

correction due to the mixing

$$\Rightarrow \quad \mathscr{R}_{nm}(\tau, L) = \left(r_{nm} + C_{nm} L^{y_t} \delta\tau + O(\delta\tau^2)\right) \left(1 + \boxed{D_{nm} L^{2\bar{y}} + O(L^{4\bar{y}})}\right)$$

$$\mathscr{B}_4(t, L) = \left(b_4 + c_4 L^{y_t} t + O(t^2)\right) \left(1 + \boxed{d_4 L^{\bar{y}} + O(L^{2\bar{y}})}\right)$$

$\bar{y} \equiv y_t - y_h \approx -0.89 < 0$ for the Ising universality

- LYZR has **faster suppression of the mixing-correction** than B_4.

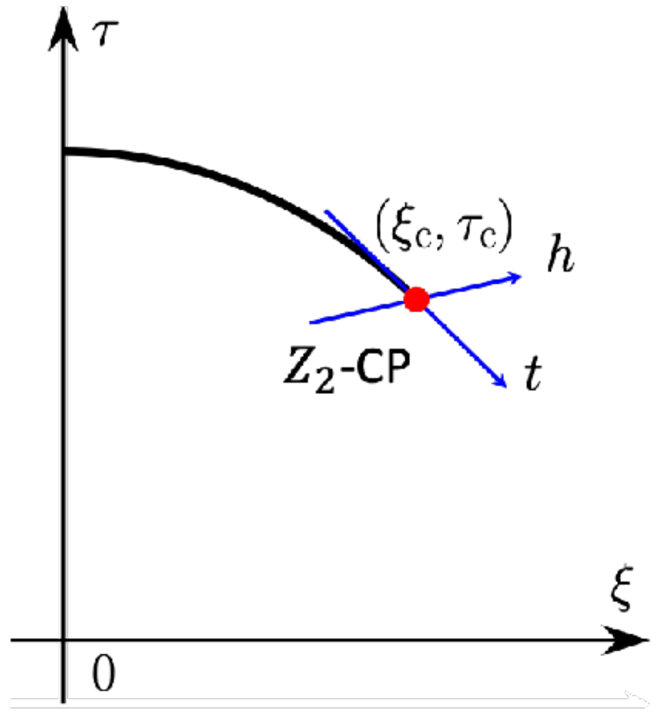
Lee-Yang Zero Ratio

[Wada-Kitazawa-KK, PRL 134, 162302 (2025)]

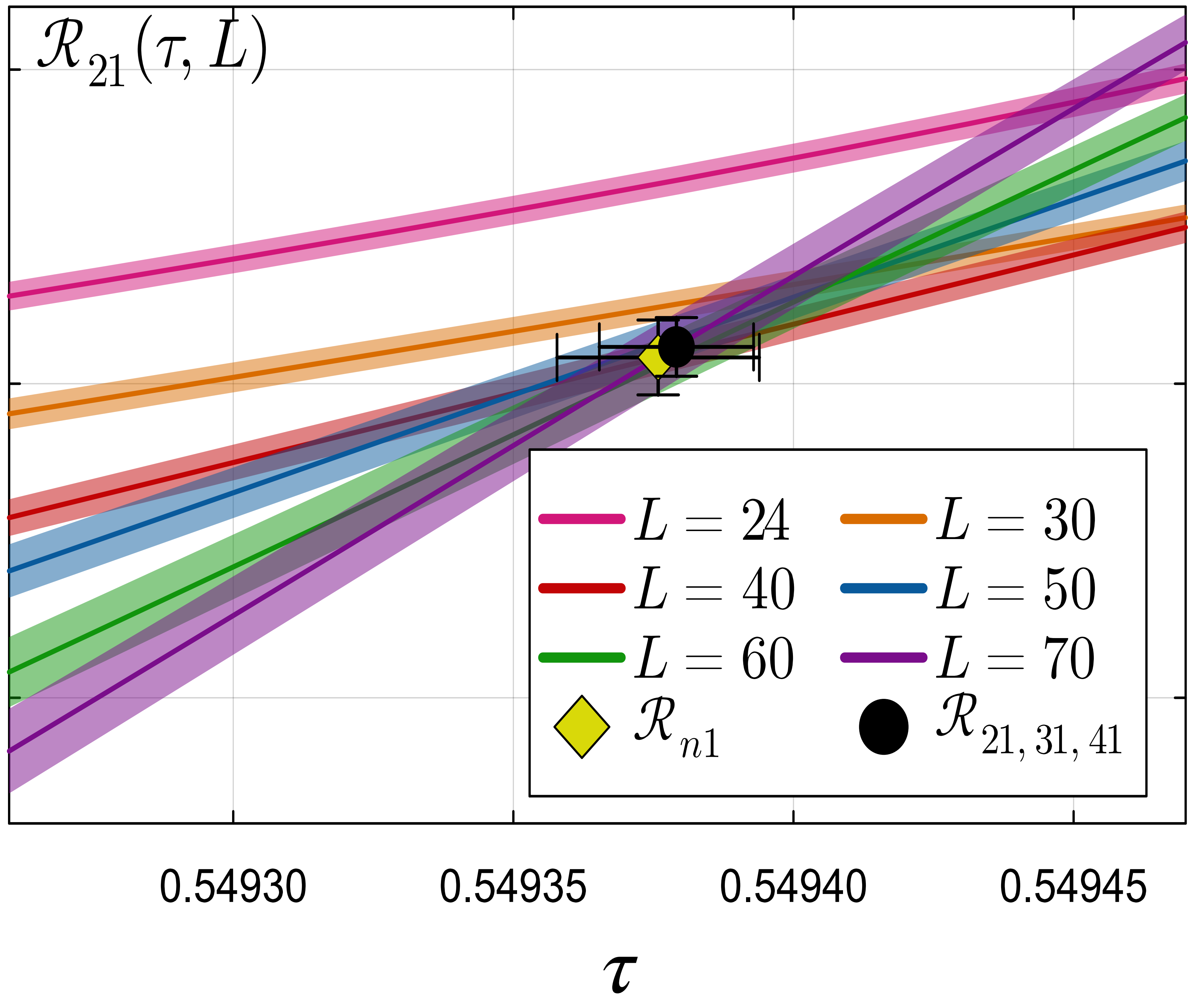
3d-Potts

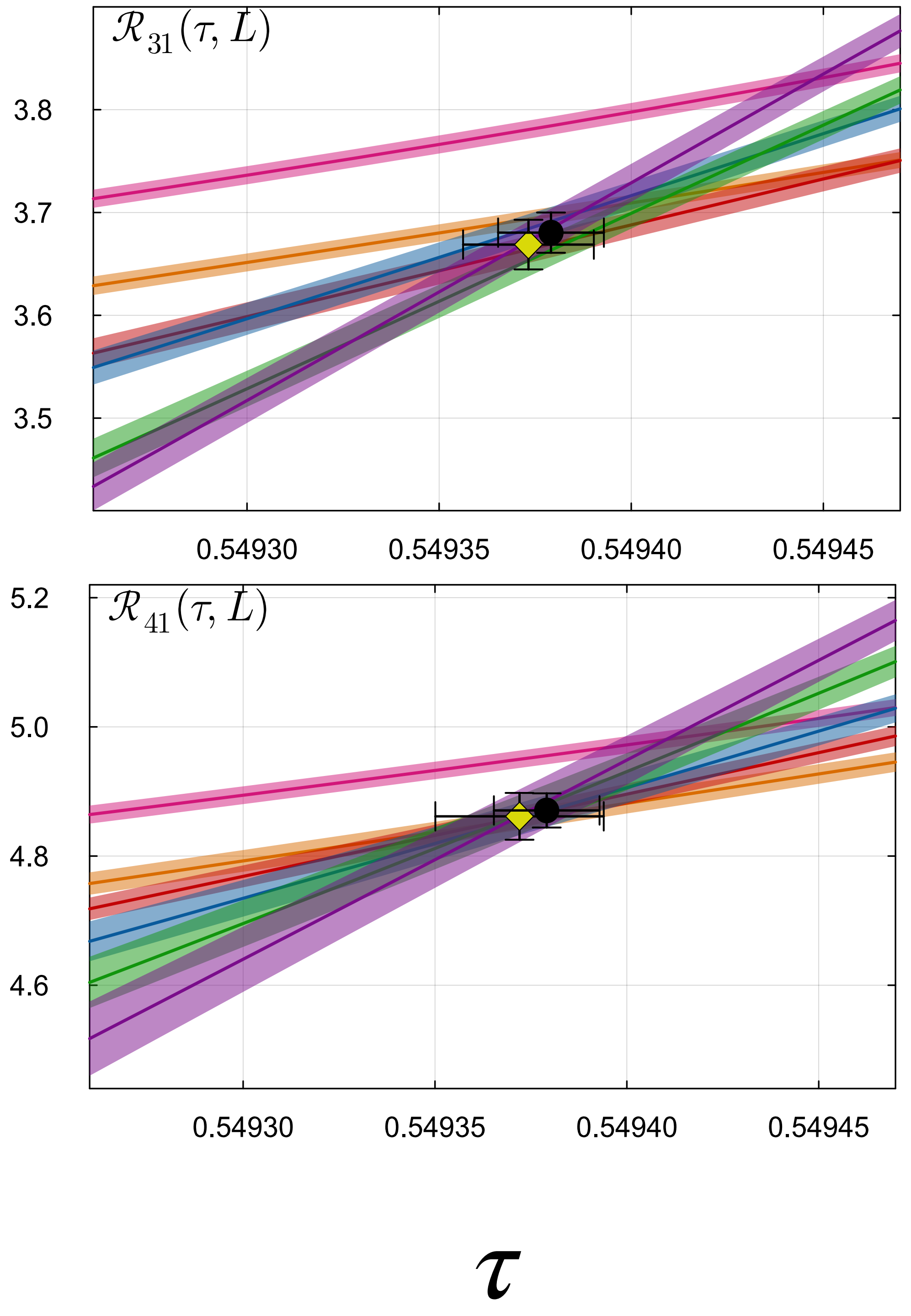
$$Z = \sum_{\{\sigma_i\}} \exp\left\{ -\tau \sum_{\langle i,j \rangle} \delta_{\sigma_i,\sigma_j} - \xi \sum_i \delta_{\sigma_i,1} \right\}$$

$\sigma_i = 1,2,3$



even-odd HB + reweighting
$L = 24,30,40,50,60,70$
3 ξ values around ξ_c
each 10^6 meas. every 10 HB steps





similar to Ising

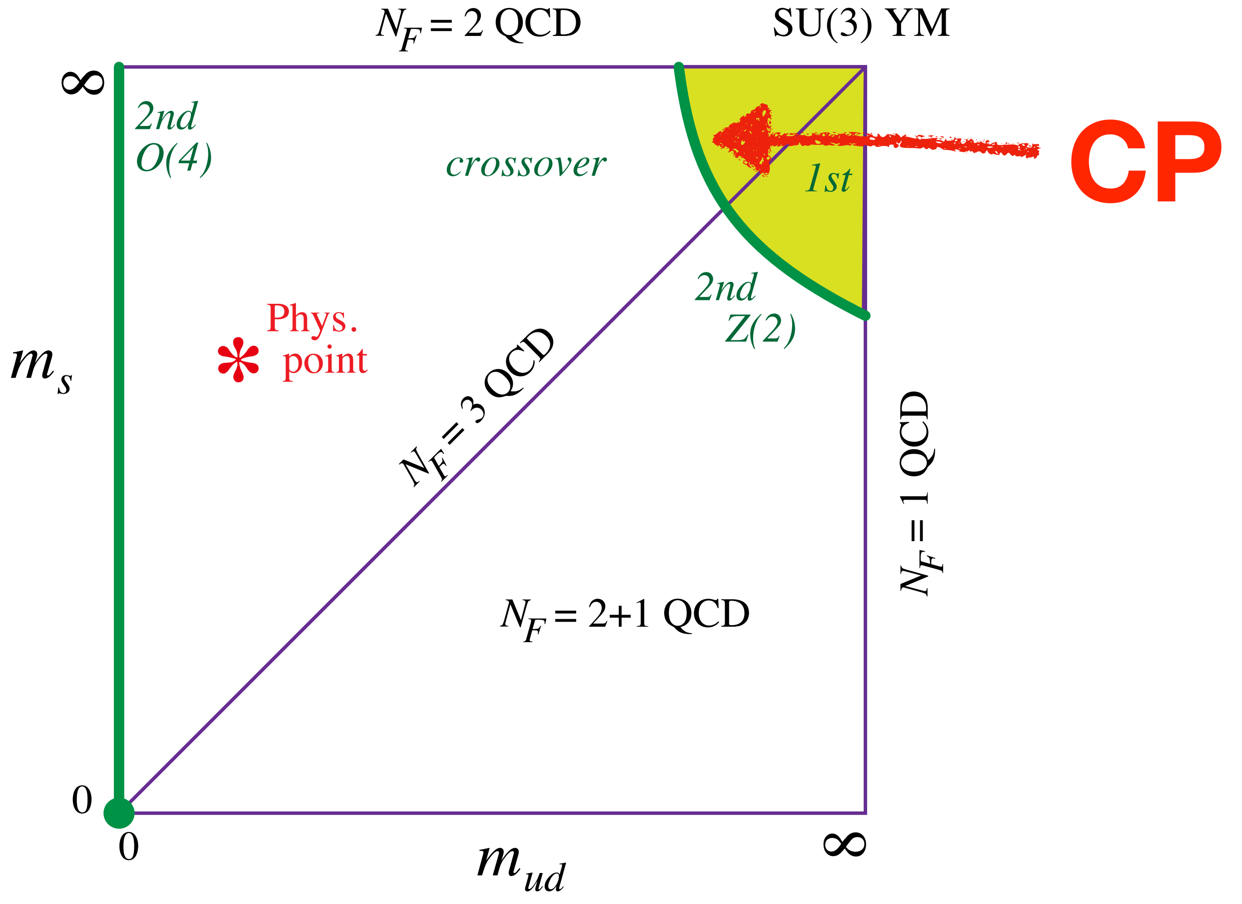
	τ_c
R_21	0.549375(18)
B_4 [our]	0.549382(11)
B_4 [Karsch (2000)]	0.54938(2)

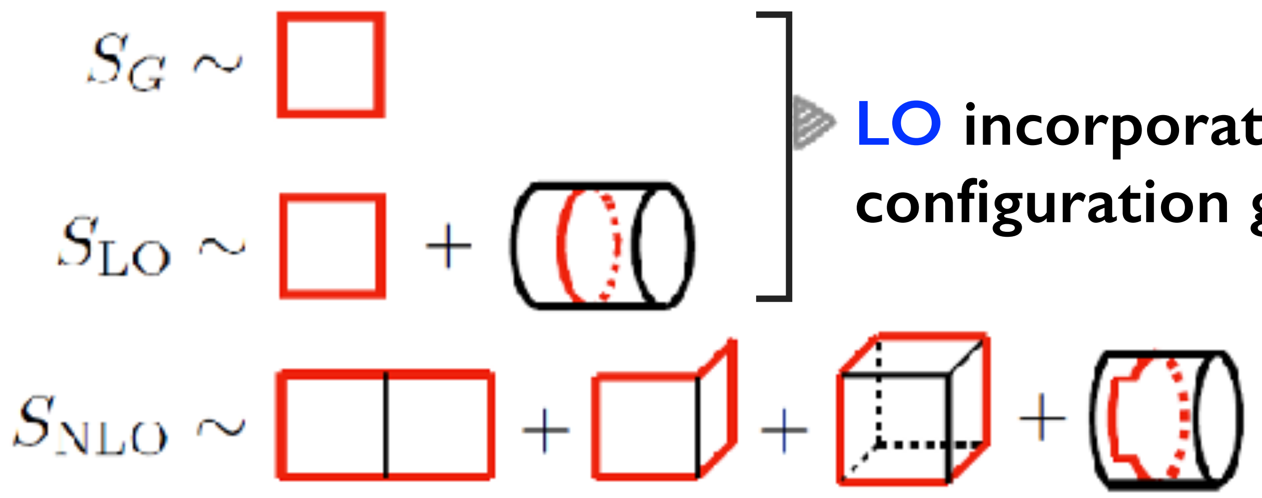
consistent with Binder

LYZR as powerful as B_4

QCD in heavy-quark region

Hopping-parameter expansion to NLO





LO incorporated in the configuration generation

NLO incorporated exactly by **reweighting**

Polyakov loop

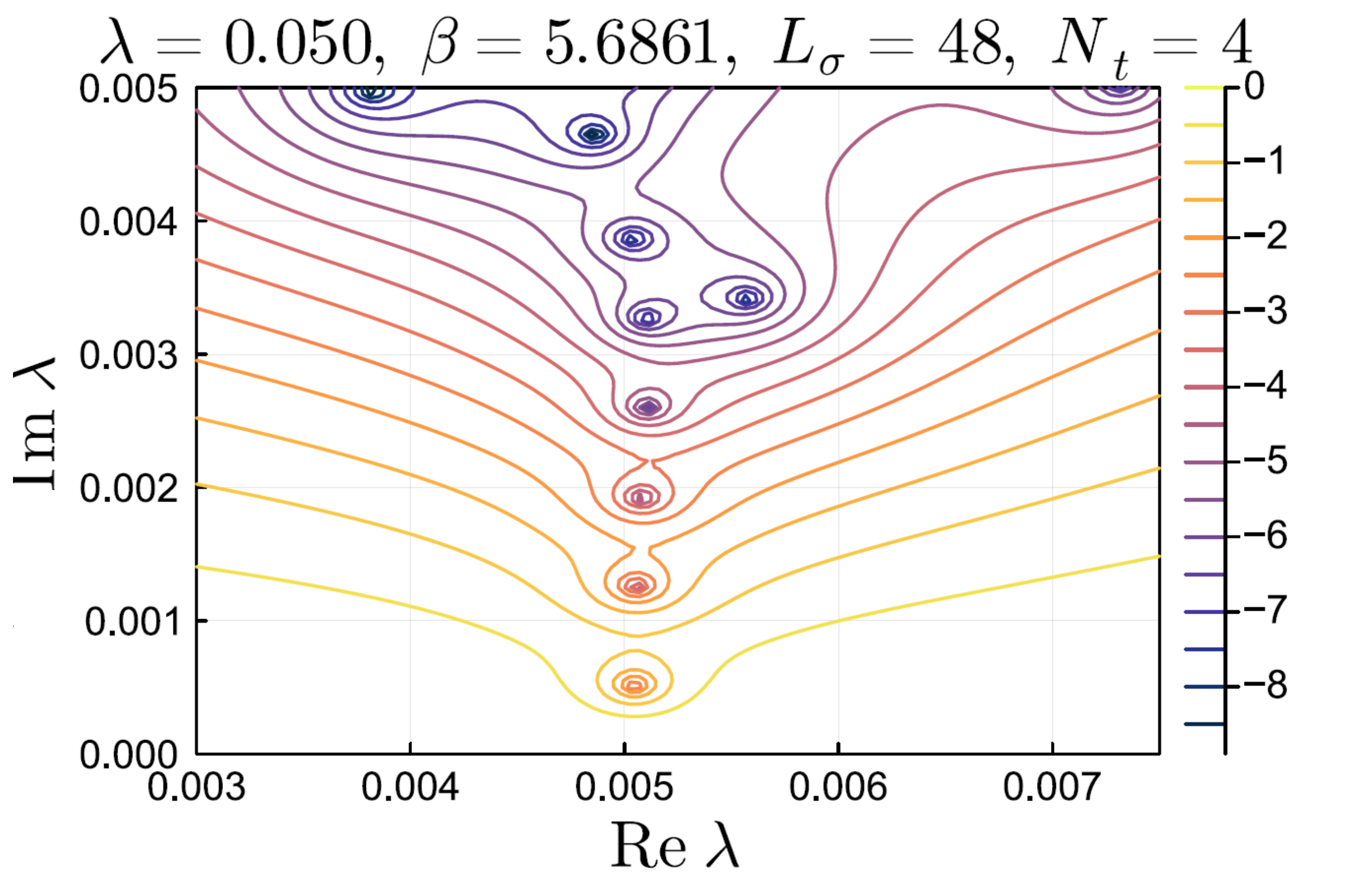
$$S_{\rm G+LO} = -6N_tN_s^3\beta^*\,\hat{P} - N_s^3\lambda\,\hat{\Omega}_{\rm R}$$

$$\beta^* = \beta + 48N_f\kappa^4, \quad \lambda(\kappa) = 48N_fN_t\kappa^4 \text{ and } 128N_fN_t\kappa^6 \text{ for } N_t = 4 \text{ and } 6$$

$\lambda\Omega_{\rm R}$ simulated by PHB+OR *à la* quenched QCD [Kiyohara+, PRD 104 (2021)]

- ☑ Accuracy of NLO-HPE confirmed up to $\kappa_c(N_t = 6)$. [Wakabayashi+, PTEP (2022]
- ☑ Effective method developed to incorporate higher orders. [Ashikawa+, PRD 110 (2024)]

Contour plot of $|Z(\lambda)/Z(\mathrm{Re}\lambda)|$

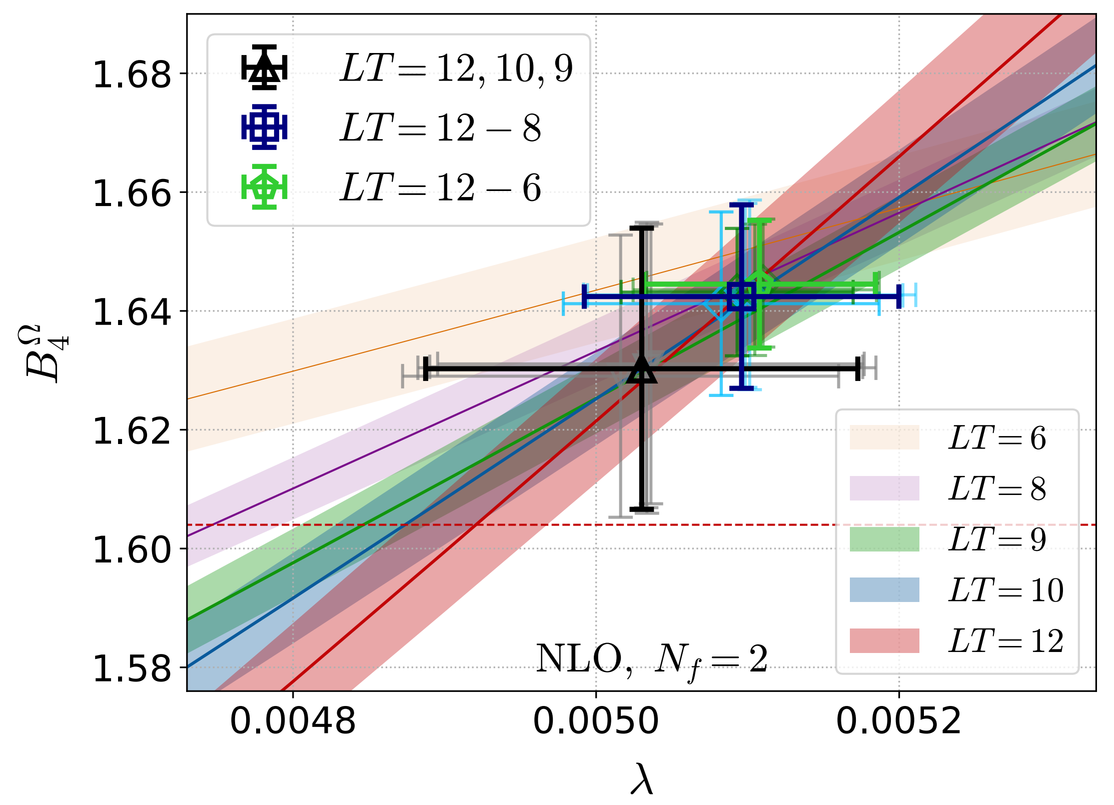


Large lattices & high stat. enabled.

⇒ precision studies of Binder-cumulant

⇒ LYZs up to 4th order available

[Ejiri, PRD 73, 054502 (2006)]

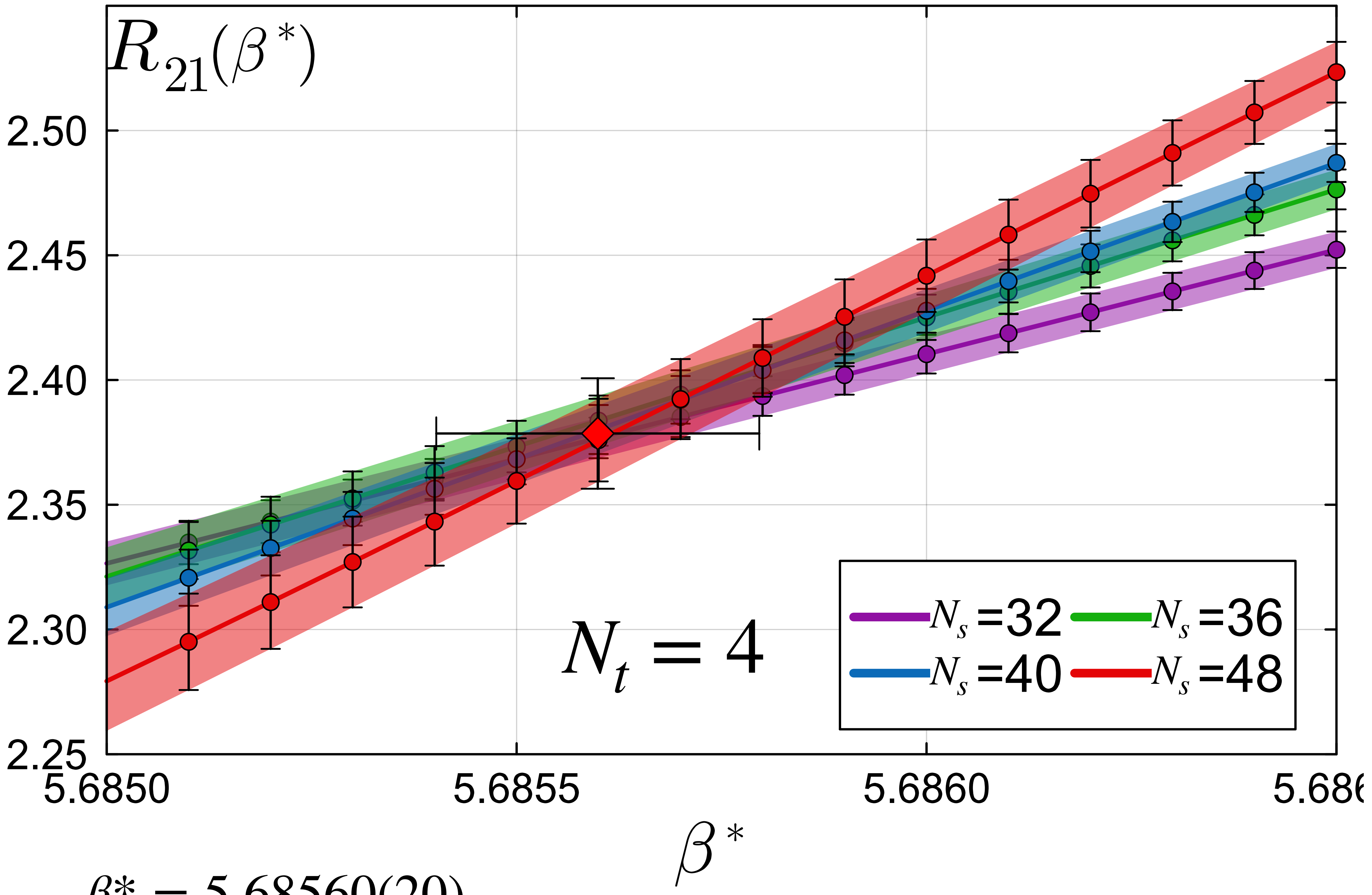


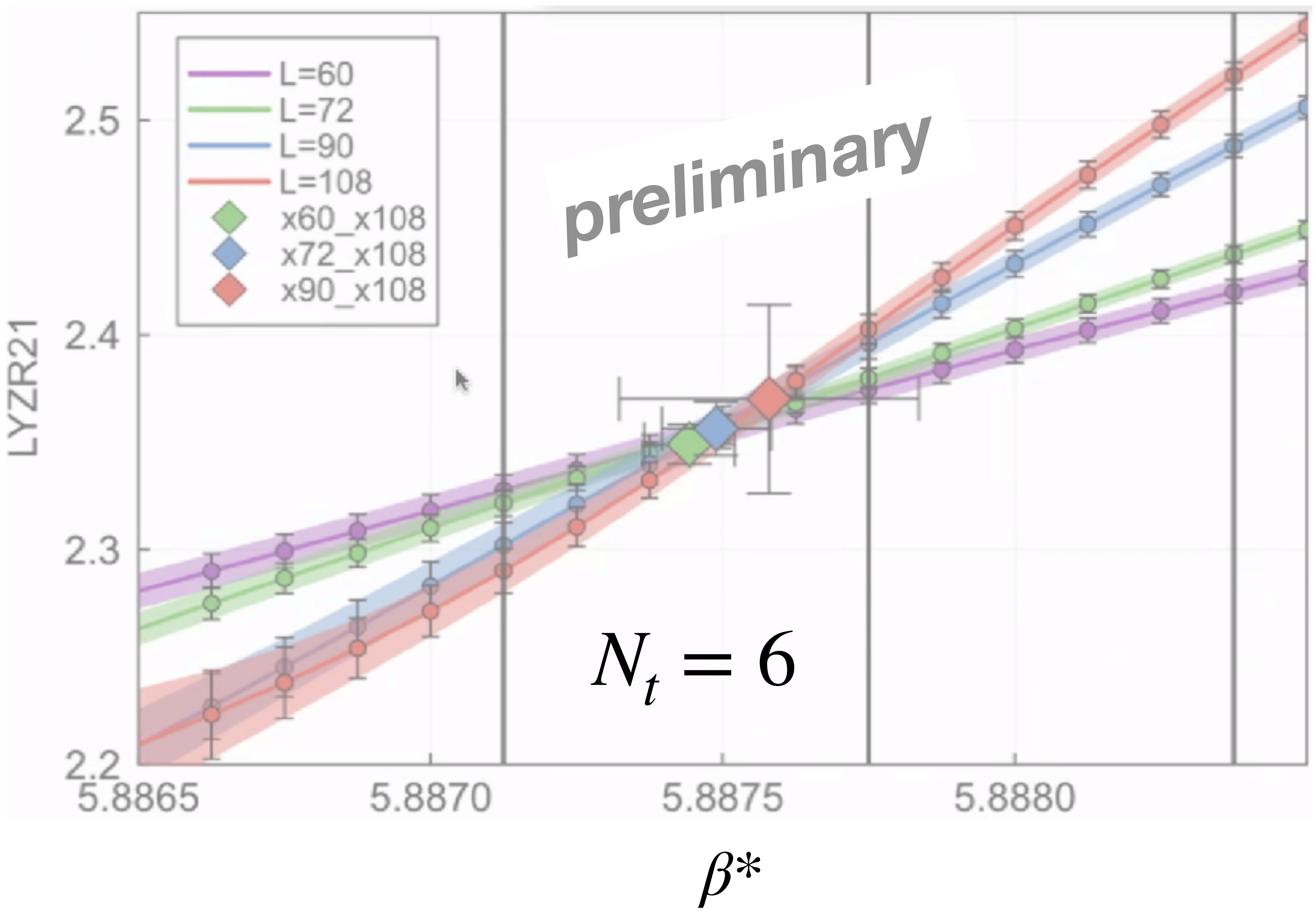
Lee-Yang Zero Ratio

[Wada-Kitazawa-KK, PoS (LATTICE 2024) 167; *on-going*]

hqQCD $S_{\mathrm{G+LO}} = -6N_tN_s^3\beta^*\hat{P} - N_s^3\lambda\,\hat{\Omega}_{\mathrm{R}}$

HPE NLO, PHB-OR + reweighting
$N_t = 4,[6],\ LT = N_s/N_t = 6,\cdots,12\,[\cdots,18]$
each $(5 \text{ to } 10)\times 10^5$ meas. every 5 PHB-OR steps

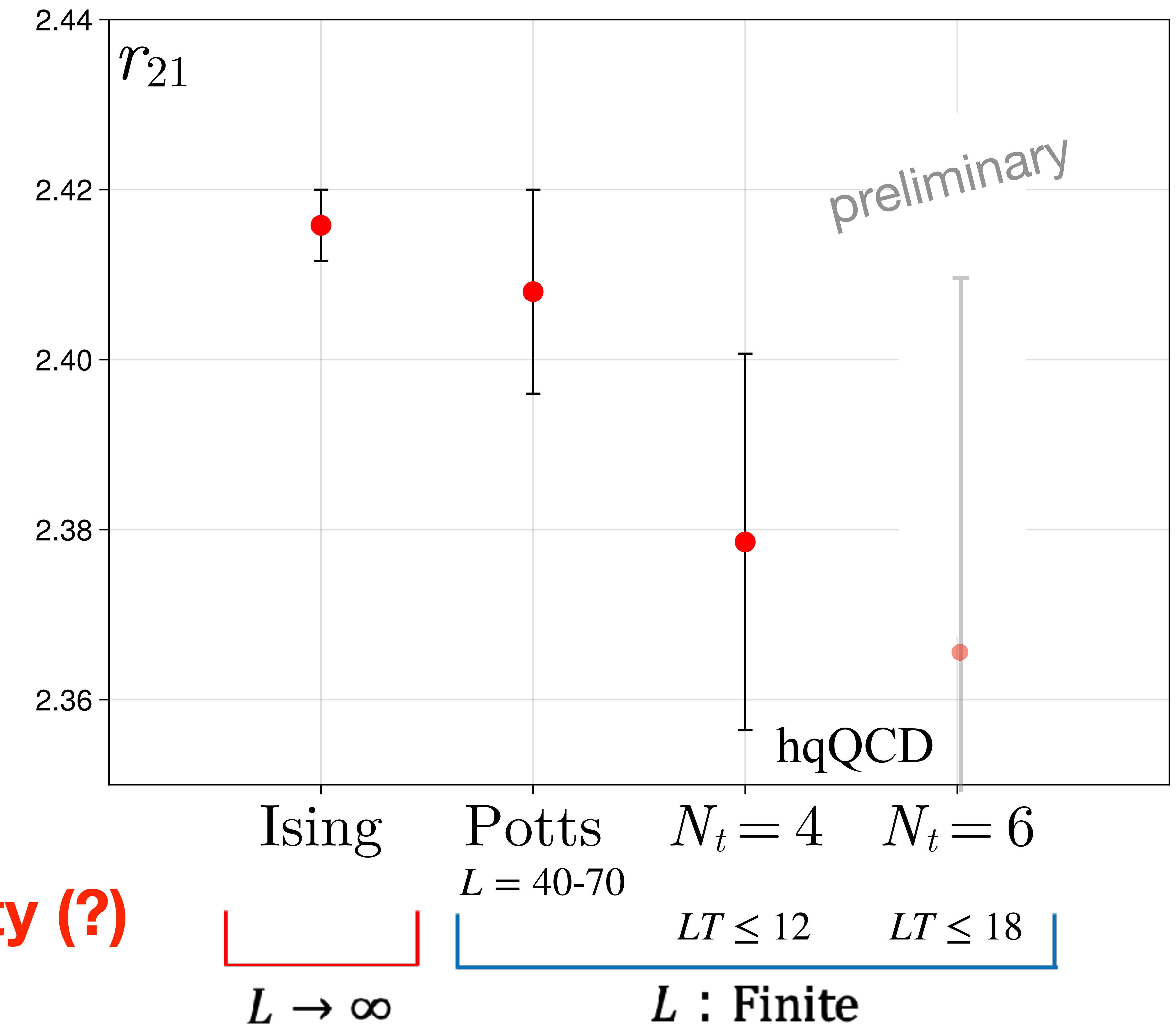




$\beta_c^* = 5.68560(20)$
cf.) $5.68579(26)$ by the Binder cumulant method.

⭐ **consistent with Binder**

Universality of r_{21}



Taking uncertainties due to differences in simulations/analyses into account,

⭐ **~consistent with Z_2 universality (?)**

more studies in hq-QCD needed.

Summary

Lee-Yang Zero Ratio: a new & independent tool to study CP

⇒ intersection analyses *à la* Binder-cumulant.

Test study of LYZR

in 3d-Ising, 3d-Potts, and heavy-quark QCD (on-going)

- ☆ **LYZR gives CP consistent with known results.**
- ☆ **LYZR is as powerful as Binder-cumulant.**
- ☆ **New universal constants r_{nm} determined.**

LYZR vs. Binder

LYZR has

- **smaller sub-leading contrib. in FSS**
- **smaller FSS-corrections**

⇒ **LYZR works with smaller L's**

- **faster suppression of the mixing-corrections** in general systems**.**

Application to heavy-quark QCD in progress.